A "CONVERSE" OF THE BANACH CONTRACTION

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Fixed-points of functions and of operators are of importance in many mathematical areas. They can, for example, be used to give precise mathematical meanings, called denotational semantics, to computer programs in many different programming paradigms. To this end, semantical operators are associated with various constructs[1] in a program in such a way that the fixed points of these operators correspond to the denotational reading of the constructs. A particular case in point is that of the semantics of logic programming systems. In this context, an operator F can often be assigned to each program P in the system in such a manner that the denotational or declarative semantics of the entire program P can be viewed in terms of the fixed points of F.

Of course, fixed-point theorems are of major importance anywhere that fixed points are important. In the theory of programming languages, the theorems used are often based on order as is the case, for example, with the well-known theorems of Knaster-Tarski and Kleene. Nevertheless, in recent years there has been a significant movement towards the use of theorems, such as the Banach contraction mapping theorem, which apply to metric spaces and to variants and generalizations of metric spaces. This is particularly so in the case of logic programming, where the presence of negation may restrict the use of order-theory.

If the Banach contraction mapping theorem is applied in order to show that a function f has a unique fixed point a, then the proof of this theorem demonstrates that the iterates of f, or the sequences $(f^n(x))_n$, converge to a for each choice of x. This strong property may allow one to compute approximations to the desired fixed point. In the context of logic programming, there are programs whose semantical operators exhibit the same behaviour in that iterates always converge to the unique fixed point, although whether or not there is a metric space underlying the process may not be immediately clear.

In this paper, we show that such a metric space can always be found, by proving the following "converse" of the Banach contraction mapping theorem.

Theorem Let (X, τ) be a topological space and let $f : X \to X$ be a function which has a unique fixed point a and is such that, for each $x \in X$, the sequence $f^n(x)$ converges to a in τ. Then there exists a function $d : X \times X \to R$ for which (X, d) is a complete ultrametric space and is such that for all $x, y \in X$ we have $d(f(x), f(y)) \leq \frac{1}{2} d(x, y)$.

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[1] In particular, those relating to recursion and self reference.

We will also present some further properties concerning the nature of the resulting metric space.

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