AXIOMATIZING STANDARD ALGEBRAS FOR FUZZY PC

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A t-norm is a binary operation $*$ (defined on the real unit interval) which is commutative, associative, non-decreasing in both arguments, and satisfies the boundary conditions $0 * x = 0$ and $1 * x = x$. In fuzzy propositional logic continuous t-norms are often taken as truth functions of the conjunction &. Together with its residuum (which is an operation $\Rightarrow$ defined by $x \Rightarrow y = \max\{z; x * z \leq y\}$, used as the truth function of the implication $\rightarrow$), each continuous t-norm $*$ yields a structure called standard algebra, denoted by $[0,1]_*$. These structures provide truth functions for the basic connectives of our version of fuzzy propositional calculus (these are &, $\rightarrow$, and a truth constant $\overline{0}$, other connectives are defined). Each continuous t-norm therefore determines a semantics for propositional logic.

In [1], three outstanding logics have been investigated in detail. These logics are given by the Łukasiewicz's t-norm ($x *_L y = \max(0, x + y - 1)$), the Gödel's t-norm ($x *_G y = \min(x, y)$), and the product t-norm ($x *_\Pi y = x.y$, multiplication) respectively. According to the representation theorem, each standard algebra is an ordinal sum of isomorphic copies of the three standard algebras given by these t-norms. For an extensive treatment of the topic please see [1].

Our aim is to investigate a standard algebra denoted $[0,1]_{L\oplus\Pi}$, which we hope to be self-explaining; it is an ordinal sum of an isomorphic copy of the Łukasiewicz's standard algebra and an isomorphic copy of the product standard algebra. These two copies are delimited by a single idempotent element h (for 'half'). We assume the logical point of view and try to find a complete axiomatization of the logic which this algebra determines.

The axiomatics presented will be an extension of a set of axioms called BL, which axiomatize tautologies of all standard algebras. To ease the task we have enriched the language of fuzzy PC with a truth constant $\overline{h}$, which is always evaluated by h. Our axioms first determine the behaviour of this constant, stating that the corresponding element of the algebra is idempotent and nonzero. This yields, by the way, quite an interesting logic in itself.

We further introduce two simple translation functions, operating on formulas, which we apply to the schematic extensions axiomatizing Łukasiewicz's and product logic respectively. We claim that the resulting sets of formulas determine the behaviour of the t-norm on the intervals $[0, h]$ and $[h, 1]$. We use the functions for this special case only, but the translation process is uniform and we suppose that analogical functions will produce schematic extensions of BL for finite numbers of intervals in the ordinal sum.

A completeness theorem w.r.t. the corresponding subvariety of BL-algebras is immediately obtained from [1]. We further prove standard completeness for our logic, that is, we show that all from the axioms.

References

[1] Hájek Petr: *Metamathematics of Fuzzy Logic*, Kluwer Academic Publishers, 1998.