DISLOCATED TOPOLOGIES

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We study a generalized notion of topology which evolved from our recent research in the area of logic programming semantics. The generalization is obtained by relaxing the requirement that a neighbourhood of a point contains the point itself (i.e. the neighbourhood is *dislocated*), and by allowing neighbourhoods of points to be empty. The corresponding generalized notion of *dislocated metric* is obtained by allowing points to have non-zero distance between themselves.

The presentation is motivated by a generalization of the Banach contraction mapping theorem, due to Matthews (1985), which says that *if f is a contraction mapping on a complete dislocated metric space, then f has a unique fixed point.* This theorem has applications in theoretical computer science in that it allows one to determine the semantics, i.e. a distinguished model, of certain logic programs, and also has applications to conventional programming language semantics. The question naturally arises as to what extent dislocated metrics and neighbourhoods can be understood from a topological point of view – a task which was undertaken by Matthews (1992) from a different perspective, and in the case of a slightly less general version of metric than is employed here, called a *partial metric*; Mathews has also shown that these spaces can be used to generate Alexandroff topologies and other structures of interest in modelling computation.

As it turns out, it is possible to define dislocated versions of convergence and continuity. When this is done, the relationship between these notions, dislocated metrics and neighbourhoods, and the generalized Banach theorem on the one hand is analogous to the relationship between the corresponding classical concepts known from elementary topology on the other. A deeper and more intuitive understanding of the spaces involved and the application mentioned above is thereby obtained.

The results and the application stated suggest that a special rôle is played by the function obtained by mapping each point in a dislocated metric space to the distance it has from itself; we call this function the *dislocation function*. Indeed, it is possible to obtain dislocated metrics via fairly general constructions involving a metric in the usual sense and a function, which then turns out to be the corresponding dislocation function. Completeness of the metric also carries over when given certain natural conditions involving continuity. Under certain other natural conditions it is also possible to obtain (pseudo-)metrics from dislocated metrics, and this again involves dislocation functions.

The very general nature of the construction of dislocated metrics from conventional metrics hints at the possibility of finding further applications of the generalized version of the Banach theorem in the areas of logic programming and artificial intelligence.

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