1 $\mathcal{B} = \{(2,\ -1,\ 2)\,,\ (2,\ -1,\ 1)\,,\ (-1,\ 0,\ -2)\}$ je báza a $\mathcal{E}$ je štandardná báza v R^3.
$T\colon R^3 \to R^3$, $T(x_1, x_2, x_3) = (x_1 + x_2\,,\ x_2 + x_3\,,\ x_1 + x_2 - x_3)$

a) [2] Napíšte maticu $[T]_{\mathcal{E}}$ lineárneho operátora T vzhľadom na štandardnú bázu.

b) [8] Napíšte maticu $[T]_{\mathcal{B}}$ lineárneho operátora T vzhľadom na bázu $\mathcal{B}$

a) $[T]_{\mathcal{E}} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$

b) $T\mathbf{b}_1 = T(2,-1,2) = (1,1-1)$, $T\mathbf{b}_2 = T(2,-1,1) = (1,0,0)$, $T\mathbf{b}_3 = T(-1,0,-2) = (-1,-2,1)$

$$\left(\begin{array}{rrr|rrr} 2 & 2 & -1 & 1 & 1 & -1 \\ -1 & -1 & 0 & 1 & 0 & -2 \\ 2 & 1 & -2 & -1 & 0 & 1 \end{array}\right) \sim_{\substack{r_1+2r_2 \\ r_3+2r_2}} \left(\begin{array}{rrr|rrr} 0 & 0 & -1 & 3 & 1 & -5 \\ -1 & -1 & 0 & 1 & 0 & -2 \\ 0 & -1 & -2 & 1 & 0 & -3 \end{array}\right) \sim \left(\begin{array}{rrr|rrr} 1 & 1 & 0 & -1 & 0 & 2 \\ 0 & 1 & 2 & -1 & 0 & 3 \\ 0 & 0 & 1 & -3 & -1 & 5 \end{array}\right) \sim$$

$$\left(\begin{array}{rrr|rrr} 1 & 1 & 0 & -1 & 0 & 2 \\ 0 & 1 & 0 & 5 & 2 & -7 \\ 0 & 0 & 1 & -3 & -1 & 5 \end{array}\right) \sim \left(\begin{array}{rrr|rrr} 1 & 0 & 0 & -6 & -2 & 9 \\ 0 & 1 & 0 & 5 & 2 & -7 \\ 0 & 0 & 1 & -3 & -1 & 5 \end{array}\right),\ \underline{T_{\mathcal{B}} = \begin{pmatrix} -6 & -2 & 9 \\ 5 & 2 & -7 \\ -3 & -1 & 5 \end{pmatrix}}$$

2. [10] Hermitovou metódou nájdite všetky celočíselné riešenia sústavy

$$x_1 + 2x_2 + x_3 - 3x_4 = 1$$
$$x_1 + 4x_2 + 2x_3 + 2x_4 = -1$$

$$\begin{pmatrix} 1 & 2 & 1 & -3 \\ 1 & 4 & 2 & 2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \sim_{\substack{s_2-2s_1 \\ s_3-s_1 \\ s_4+3s_1}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 5 \\ 1 & -2 & -1 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \sim_{s_2 \leftrightarrow s_3} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 5 \\ 1 & -1 & -2 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \sim_{\substack{s_3-2s_2 \\ s_4-5x_2}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 8 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -2 & -5 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} 1 & -1 & 0 & 8 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -2 & -5 \\ 0 & 0 & 0 & 1 \end{pmatrix},\quad \begin{array}{c} y_1 = 1 \\ y_1 + y_2 = -1 \end{array} \qquad \mathbf{y} = \begin{pmatrix} 1 \\ -2 \\ a \\ b \end{pmatrix}$$

$$U\mathbf{y} = \begin{pmatrix} 1 & -1 & 0 & 8 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -2 & -5 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ a \\ b \end{pmatrix} = \begin{pmatrix} 3 + 8b \\ a \\ -2 - 2a - 5b \\ b \end{pmatrix} \quad \underline{P = \{(3 + 8b\,,\ a\,, -2 - 2a - 5b\,,\ b)\colon a, b \in Z\}}$$

3. [10] $A = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{pmatrix}$.

a) [2] Napíšte stopu trace(A)

b) [3] Nájdite vlastné čísla matice A

c) [5] Určte diagonálnu maticu D a maticu $P \in R^{n\times n}$ tak, aby $A = PDP^\top$

a) $\text{trace}(A) = 4 + 2 + 2 = 8$

b) Matica A má dva rovnaké riadky $\implies \det A = 0 \implies \underline{\lambda_1 = 0}$,

$\det(A - \lambda I) = (4-\lambda)\begin{vmatrix} 2-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix} \implies \underline{\lambda_2 = 4}$, $8 = 0 + 4 + \lambda_3 \implies \underline{\lambda_3 = 4}$

c) Hľadáme 3 navzájom kolmé vlastné vektory matice A dĺžky 1:

$\lambda_1 = 0$, $\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, $\|\mathbf{v}_1\| = \sqrt{2}$, $P_{*1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

d) $\lambda_{2,3} = 4$, $\begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 2 & -2 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $\|\mathbf{v}_3\| = \sqrt{2}$

$$\underline{P = \begin{pmatrix} 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}} \quad \underline{D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}}$$

4 $A = \begin{pmatrix} 5 & -5 & -3 \\ 3 & -3 & -2 \\ 2 & -2 & -1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. Určte

a) [1] trace(A),

b) [3] minimálny polynóm $m_{A,\mathbf{b}}(\lambda)$ vektora $\mathbf{b}$ vzhľadom na maticu A.

c) [3] vlastné čísla matice A,

d) [6] Jordanovu maticu J a maticu P tak, aby $A = PJP^{-1}$.

a) $\underline{\text{trace}(A) = 1}$

b) Vypočítame $A^0\mathbf{b} = \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $A^1\mathbf{b} = A\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$, $A^2\mathbf{b} = A(A\mathbf{b}) = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = A^3\mathbf{b}$

a riešime sústavu $a_0\mathbf{b} + a_1(A\mathbf{b}) + a_2(A^2\mathbf{b}) + a_3a_2(A^3\mathbf{b}) = (0 \quad 0 \quad 0)^\top$:

$$\begin{pmatrix} 1 & 2 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{matrix} a_0 = 0 \\ a_3 = 0, \quad a_2 = 1 \\ a_1 = -1 \end{matrix}$$

$\underline{m_{A,\mathbf{b}}(\lambda) = \lambda^2 - \lambda = \lambda(\lambda - 1)}$

c) $\lambda_1 = 1$, $\lambda_{2,3} = 0$

d) Vl. vektory $\lambda = 1$ (netreba hľadať, stačí si všimnúť, že v b) sme ukázali $A(A\mathbf{b}) = A\mathbf{b}$:

$$\begin{pmatrix} 4 & -5 & -3 \\ 3 & -4 & -2 \\ 2 & -2 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \; P_{*1} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

$$\lambda = 0\colon \begin{pmatrix} 5 & -5 & -3 \\ 3 & -3 & -2 \\ 2 & -2 & -1 \end{pmatrix} \sim_{r_1 - 2r_3} \begin{pmatrix} 1 & -1 & -1 \\ 3 & -3 & -2 \\ 2 & -2 & -1 \end{pmatrix} \sim_{\substack{r_2 - 3r_1 \\ r_3 - 2r_1}} \begin{pmatrix} 1 & -1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \; P_{*3} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

a potrebujeme ešte zovšeobecnený vl. vektor

$$\left(\begin{array}{ccc|c} 5 & -5 & -3 & 1 \\ 3 & -3 & -2 & 1 \\ 2 & -2 & -1 & 0 \end{array}\right) \sim_{r_1 - 2r_3} \begin{pmatrix} 1 & -1 & -1 & 1 \\ 3 & -3 & -2 & 1 \\ 2 & -2 & -1 & 0 \end{pmatrix} \sim_{\substack{r_2 - 3r_1 \\ r_3 - 2r_1}} \begin{pmatrix} 1 & -1 & -1 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & -2 \end{pmatrix}$$

$$\begin{matrix} x_3 = -2, \\ x_1 - x_2 + 2 = 1, \\ x_2 = x_1 + 1 \end{matrix} \text{ napr. } P_{*2} = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$$

$$\underline{J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & -2 & 0 \end{pmatrix}}$$

5. Nech $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4\}$ je báza LP $(L, +, \cdot)$ a $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2\}$ je báza LP $(M, +, \cdot)$ nad $\mathbb{C}$. $T\colon L \to M$ je lineárny operátor, $T\mathbf{b}_1 = (1-i)\mathbf{d}_1 + \mathbf{d}_2$, $T\mathbf{b}_2 = (1-i)\mathbf{d}_1 + \mathbf{d}_2$, $T\mathbf{b}_3 = \mathbf{d}_1$, $T\mathbf{b}_4 = \mathbf{d}_2$. Napíšte

a) [1] $\dim M$, b) [3] maticu $T_{\mathcal{BD}}$,

c) [4] $\dim \operatorname{Ker} T$, $\dim \operatorname{Ran} T$.

a) $\dim M = 2$, b) $T_{\mathcal{BD}} = \begin{pmatrix} 1-i & 1-i & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$

c) $\operatorname{Ker} T = \{(a, b, (i-1)a + (i-1)b, -a-b)\colon a, b \in R\} \implies \underline{\dim \operatorname{Ker}(T) = 2}$, $\underline{\dim \operatorname{Ran}(T) = 4 - \dim \operatorname{Ker}(T) = 2}$.

6. [10] Nech $f_1, f_2 \in P(\mathbb{R})$, $f_1(x) = 4x^3 - 2x^2 + 1$, $f_2(x) = 2x^2 - 3x - 2$. Určte $a_1(x), a_2(x) \in P(\mathbb{R})$, pre ktoré $\gcd(f_1, f_2) = a_1(x)f_1(x) + a_2(x)f_2(x)$.

$$
\begin{aligned}
&(4x^3 - 2x^2 + 1) : (2x^2 - 3x - 2) = 2x + 2 \\
&-\underline{(4x^3 - 6x^2 - 4x)} \\
&\qquad 4x^2 + 4x + 1 \\
&\quad -\underline{(4x^2 - 6x - 4)} \\
&\qquad\quad 10x + 5 = f_3, \quad f_1 = (2x+2)f_2 + f_3 \\
&(2x^2 - 3x - 2) : (10x + 5) = \frac{1}{5}x - \frac{2}{5} \\
&-\underline{(2x^2 + x)} \\
&\qquad -4x - 2 \\
&\quad -\underline{(-4x - 2)} \\
&\qquad\quad 0 \implies \gcd(f_1, f_2) = f_3 = f_1 - (2x+2)f_2, \quad \underline{a_1 = 1,\ a_2 = -(2x+2)}
\end{aligned}
$$

7. [8] $A \in \mathbb{R}^{3\times 5}$, $A = \begin{pmatrix} 1 & 0 & 2 & 2 & 1 \\ 1 & 2 & 2 & 0 & 2 \\ 1 & 1 & 2 & 2 & 1 \end{pmatrix}$

$L_s(A)$ ($L_r(A)$) je lineárny obal stĺpcov (riadkov) matice A.
Určte bázy $\mathcal{B}_s$ a $\mathcal{B}_r$ priestorov $L_s(A)$ a $L_r(A)$. Napíšte $\dim L_s(A)$, $\dim L_r(A)$.

$$
A = \begin{pmatrix} 1 & 0 & 2 & 2 & 1 \\ 1 & 2 & 2 & 0 & 2 \\ 1 & 1 & 2 & 2 & 1 \end{pmatrix} \sim_{\substack{r_2 - r_1 \\ r_3 - r_1}} \begin{pmatrix} 1 & 0 & 2 & 2 & 1 \\ 0 & 2 & 0 & -2 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 & 2 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 1 \end{pmatrix} = B
$$

$\mathcal{B}_r = \{B_{1*}, B_{2*}, B_{3*}\}$, $\mathcal{B}_s = \{A_{*1}, A_{*2}, A_{*4}\}$, $\dim L_s(A) = \dim L_r(A) = 3$

8. [7] Napíšte rovnicu priamky $y = kx + q$, pre ktorú je odchýlka $\sum_{i=-2}^{2} |y(x_i) - y_i|^2$ najmenšia.

y_i sú hodnoty namerané v bodoch x_i:

x_i	-2	-1	0	1	2
y_i	-1	$-0,6$	$-0,5$	$-0,2$	$0,9$

Priamku p aj body $[x_i, y_i]$ znázornite.

$$
\begin{aligned}
-2k + q &= -1 \\
-k + q &= -0,6 \\
q &= -0,5 \\
k + q &= -0,2 \\
2k + q &= 0,9
\end{aligned}
\qquad
A = \begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} -1 \\ -0,6 \\ -0,5 \\ -0,2 \\ 0,9 \end{pmatrix}, \ A^\top A = \begin{pmatrix} 10 & 0 \\ 0 & 5 \end{pmatrix}, \ A^\top \mathbf{b} = \begin{pmatrix} 4,2 \\ -1,4 \end{pmatrix}
$$

$$
A^\top A \begin{pmatrix} k \\ q \end{pmatrix} = \begin{pmatrix} 10 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} k \\ q \end{pmatrix} = \begin{pmatrix} 4,2 \\ -1,4 \end{pmatrix} \implies \underline{y = 0,42x - 0,28}
$$

Na obrázku má byť zrejmé, ktoré body $[x_i, y_i]$ ležia pod priamkou a ktoré nad priamkou p

9. V priestore polynómov $P(Z_2)$

   a. [3] napíšte všetky ireducibilné polynómy stupňa 2 a stupňa 3,

   b. [5] rozhodnite, či je polynóm $g(x) = x^5 + x^3 + 1$ ireducibilný.

   a. $x^2 + x + 1$, $x^3 + x + 1$, $x^3 + x^2 + 1$

   b. $g(0) = g(1) = 1 \neq 0$
   $g(x) : (x^2 + x + 1)$ nevychádza bez zvyšku, <u>je ireducibilný</u>.

10. Riešte sústavy, ich rozšírené matice najprv upravte na redukované stupňovité.

a) [5] v Z_2
$$\begin{aligned} x_1 + x_3 &= 1 \\ x_1 + x_2 + x_4 &= 0 \\ x_2 + x_3 + x_4 &= 1 \end{aligned}$$

b) [5] v Z_3
$$\begin{aligned} x_1 + 2x_2 &= 1 \\ 2x_1 + x_2 + 2x_3 &= 2 \\ x_2 + 2x_3 &= 1 \end{aligned}$$

$$a)\ \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{array}\right) \sim_{r_2+r_1} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{array}\right) \sim_{r_3+r_2} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\Longrightarrow \begin{array}{c} x_1 = 1 + a \\ x_2 = 1 + a + b \\ x_3 = a, x_4 = b; a, b \in Z_2 \end{array} \qquad \underline{P = \{(1,1,0,0); (1,0,0,1); (0,0,1,0); (0,1,1,1)\}}$$

$$b)\ \left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 2 & 1 & 2 & 2 \\ 0 & 1 & 2 & 1 \end{array}\right) \sim_{r_2+r_1} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 2 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right) \sim_{r_2+r_3} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right) \sim_{r_1+r_2}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right) \Longrightarrow \underline{P = \{2, 1, 0)\}}$$

11. [6] V poli Z_{43} vypočítajte $a = 12^{-1}$ (t.j. $a \in \mathbb{Z}$ také, že $12a = 1 \pmod{43}$, $0 < a < 43$).

43	12	3	1	0	0	1
12	7	1	0	1	1	−3
7	5	1	1	−1	−3	4
5	2	2	−1	2	4	−7
2	1		2	−5	−7	18

$\underline{a = 12^{-1} = 18}$