

(1) Riešte okrajovú úlohu

$$\cos^2 x \ u'' - \sin 2x \ u' = \cos x, \quad 0 < x < \frac{\pi}{3}, \quad u'(0) = u\left(\frac{\pi}{3}\right) = 0,$$

$$(\cos^2 x)' = -2 \sin x \cos x = -\sin 2x.$$

$$(\cos^2 x \ u')' = \cos x, \quad \cos^2 x \ u' = \sin x + c_1,$$

$$u' = \frac{\sin x}{\cos^2 x} + \frac{c_1}{\cos^2 x},$$

$$u'(0) = c_1 = 0,$$

$$(\text{subst. } \cos x = t) \quad u(x) = \frac{1}{\cos x} + c_2,$$

$$u\left(\frac{\pi}{3}\right) = 2 + c_2 = 0 \Rightarrow c_2 = 2,$$

$$u(x) = \frac{1}{\cos x} - 2.$$

(2) Riešte úlohu na vlastné hodnoty a vlastné funkcie.

$$u'' + \lambda u = 0, \quad 0 < x < 2, \quad u'(0) = u'(2) = 0.$$

$$q(x) = 0 \Rightarrow \lambda \geq 0$$

$$\text{a)} \quad \lambda = 0, \Rightarrow u(x) = c_1 + c_2 x$$

$$u'(0) = u'(2) = c_2 = 0, \quad c_1 = 1, \Rightarrow \lambda_0 = 0, \quad u_0(x) = 1,$$

$$\text{b)} \quad \lambda > 0, \quad u(x) = c_1 \cos \sqrt{\lambda} x + c_2 \sin \sqrt{\lambda} x,$$

$$u'(x) = -\sqrt{\lambda} c_1 \sin \sqrt{\lambda} x + \sqrt{\lambda} c_2 \cos \sqrt{\lambda} x,$$

$$u'(0) = \sqrt{\lambda} c_2 = 0 \Rightarrow c_2 = 0, \quad c_1 = 1, \quad u(x) = \cos \sqrt{\lambda} x,$$

$$u'(2) = -\sqrt{\lambda} \sin 2\sqrt{\lambda} = 0 \Rightarrow 2\sqrt{\lambda} = n\pi, \quad \sqrt{\lambda} = \frac{n\pi}{2}.$$

$$\lambda_n = \left(\frac{n\pi}{2}\right)^2, \quad u_n(x) = \cos \frac{n\pi}{2} x, \quad n = 0, 1, 2, \dots.$$

(3) Riešte okrajovú úlohu

$$\Delta = 0, \quad 0 < x < 2\pi, \quad 0 < y < 1,$$

$$u(0, y) = \partial_x u(2\pi, y) = u(x, 0) = 0, \quad u(x, 1) = x.$$

*Riešenie:*

$$u(x, y) = X(x)Y(y).$$

Po dosadení do rovnice:

$$X''Y + XY'' = 0 / \frac{1}{XY},$$

$$Y'' - \lambda Y(y) = 0,$$

$$\frac{Y''}{X} + \frac{Y''}{Y} = 0 \Rightarrow \frac{Y''}{X} = -\frac{Y''}{Y} = -\lambda.$$

$$X'' + \lambda X = 0, \quad X(0) = X'(2\pi) = 0.$$

Z prvej okrajovej podmienky:  $\lambda > 0, \quad X(x) = \sin \sqrt{\lambda} x,$

Z druhej okrajovej podmienky:

$$X'(2\pi) = \sqrt{\lambda} \cos(\sqrt{\lambda} 2\pi) = 0 \Rightarrow \sqrt{\lambda} 2\pi = \frac{(2n-1)\pi}{2},$$

$$\sqrt{\lambda} = \frac{2n-1}{4}, \quad \lambda \equiv \lambda_n = \left(\frac{2n-1}{4}\right)^2, \quad X \equiv X_n(x) = \sin \frac{2n-1}{4} x.$$

Dosadením do rovnice pre  $Y$ :

$$Y_n'' - \left(\frac{2n-1}{4}\right)^2 Y_n = 0, \quad Y_n(0) = 0.$$

$$Y_n(y) = a_n \cosh \frac{2n-1}{4}y + b_n \sinh \frac{2n-1}{4}y, \quad 0 < y < 1.$$

$$Y_n(0) = a_n \cosh 0 = a_n = 0.$$

$$Y_n(y) = b_n \sinh \frac{2n-1}{4}y, \quad 0 < y < 1.$$

$$u_n(x, y) = Y_n(y)X_n(x) = b_n \sinh \frac{2n-1}{4}yy \sin \frac{2n-1}{4}x.$$

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sinh \frac{2n-1}{4}y \sin \sinh \frac{2n-1}{4}x.$$

Z nenulovej okrajovej podmienky:

$$u(x, 1) = x = \sum_{n=1}^{\infty} b_n \sinh \frac{2n-1}{4} \sin \frac{2n-1}{4}x.$$

Fourierove koeficienty:

$$B_n = b_n \sinh \frac{2n-1}{4} = \frac{1}{\pi} \int_0^{2\pi} x \sin \frac{2n-1}{4}x =$$

$$\frac{1}{\pi} \int_0^{2\pi} \left(\frac{4}{2n-1}\right) \cos \frac{2n-1}{4}x dx = \frac{16(-1)^{n-1}}{(2n-1)^2 \pi},$$

$$b_n = \frac{16(-1)^{n-1}}{(2n-1)^2 \pi \sinh \frac{2n-1}{4}}.$$

*Výsledok:*

$$u(x, y) = \sum_{n=1}^{\infty} \frac{16(-1)^{n-1}}{(2n-1)^2 \pi \sinh \frac{2n-1}{4}} \sinh \frac{2n-1}{4}y \sin \frac{2n-1}{4}x.$$