

(1) Riešte okrajovú úlohu

$$\Delta = 0, \quad 0 < x < 1, \quad 0 < y < 1,$$

$$u(0, y) = u(1, y) = \frac{\partial u}{\partial y}(x, 0) = 0, \quad \frac{\partial u}{\partial y}u(x, 1) = x.$$

Riešenie:

$$u(x, y) = X(x)Y(y).$$

Po dosadení do rovnice:

$$X''Y + XY'' = 0 / \frac{1}{XY},$$

$$Y'' - \lambda Y(y) = 0,$$

$$\frac{X''}{X} + \frac{Y''}{Y} = 0 \Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda.$$

$$X'' + \lambda X = 0, \quad X(0) = X(1) = 0.$$

Z prvej okrajovej podmienky: $\lambda > 0$, $X(x) = \sin \sqrt{\lambda}x$,

Z druhej okrajovej podmienky:

$$X(1) = \sin(\sqrt{\lambda}) = 0 \Rightarrow \sqrt{\lambda} = n\pi.$$

$$\lambda \equiv \lambda_n = (n\pi)^2, \quad X \equiv X_n(x) = \sin n\pi x.$$

Dosadením do rovnice pre Y :

$$Y_n'' - (n\pi)^2 Y_n = 0, \quad Y_n'(0) = 0.$$

$$Y_n(y) = a_n \cosh n\pi y + b_n \sinh n\pi y, \quad 0 < y < 1.$$

$$Y_n'(0) = b_n n\pi \cosh 0 = 0.$$

Z okrajovej podmienky: $b_n = 0$, $Y_n(y) = a_n \cosh n\pi y$.

$$u_n(x, y) = Y_n(y)X_n(x) = a_n \cosh n\pi y \sin n\pi x.$$

$$u(x, y) = \sum_{n=1}^{\infty} a_n \cosh n\pi y \sin n\pi x.$$

Z nenulovej okrajovej podmienky:

$$\frac{\partial u}{\partial y}(x, 1) = 1 = \sum_{n=1}^{\infty} a_n n\pi \sinh n\pi \sin n\pi x.$$

Fourierove koeficienty:

$$A_n = a_n n\pi \sinh n\pi = 2 \int_0^1 x \sin n\pi x =$$

$$-\frac{2}{n\pi} \cos n\pi + \int_0^1 \frac{2}{\pi n} \cos n\pi x dx = \frac{2}{n\pi} (-1)^{n-1},$$

$$a_n = \frac{2}{n^2 \pi \sinh n\pi}.$$

Výsledok:

$$u(x, y) = \sum_{n=1}^{\infty} \frac{2(-1)^{n-1}}{n^2 \pi^2 \sinh n\pi} \cosh n\pi y \sin n\pi x.$$

(2) Riešte okrajovú úlohu

$$\begin{aligned}\Delta u(x, y) &= 0, \quad x^2 + y^2 < 4, \quad x > 0, \quad y > 0; \\ u(x, 0) &= 0, \quad \frac{\partial u}{\partial x}(0, y) = 0, \quad u(x, y) = 1, \quad \text{ak } x^2 + y^2 = 4.\end{aligned}$$

Riešenie:

$$\begin{aligned}\Delta u(r, \varphi) &= \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \varphi^2} = 0, \\ u(r, 0) &= \frac{\partial u}{\partial \varphi}(r, \frac{\pi}{2}) = 0, \quad u(2, \varphi) = 1. \\ r^2 \Delta u(r, \varphi) &= r^2 \frac{\partial^2 u}{\partial r^2} + r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \varphi^2} = 0, \quad 0 \leq r < 2, \quad 0 < \varphi < \frac{\pi}{2}.\end{aligned}$$

Dosadiť $u(r, \varphi) = R(r)\Phi(\varphi)$:

$$r^2 R'' \Phi + r R' \Phi + R \Phi'' = 0 / \frac{1}{R \Phi},$$

$$\frac{r^2 R'' + r R'}{R} = -\frac{\Phi''}{\Phi} = \lambda.$$

$$r^2 R'' + r R' - \lambda R(r) = 0, \quad 0 \leq r < 2,$$

$$\Phi'' + \lambda \Phi = 0, \quad \Phi(0) = \Phi'(\frac{\pi}{2}) = 0.$$

Z prvej okrajovej podmienky: $\lambda > 0, \Phi(\varphi) = \sin \sqrt{\lambda} \varphi$.

Z druhej okrajovej podmienky:

$$\begin{aligned}\Phi'(\frac{\pi}{2}) &= \sqrt{\lambda} \cos \sqrt{\lambda} \frac{\pi}{2} = 0, \Rightarrow \sqrt{\lambda} \frac{\pi}{2} = (2n-1) \frac{\pi}{2}, \quad n = 1, 2, \dots \\ \lambda &\equiv \lambda_n = (2n-1)^2, \quad \Phi \equiv \Phi_n(\varphi) = \sin((2n-1)\varphi).\end{aligned}$$

$$r^2 R_n'' + r R_n' - (2n-1)^2 R_n(r) = 0, \quad 0 \leq r < 2,$$

$$R_n(r) = a_n r^{2n-1} + b_n r^{-(2n-1)},$$

$b_n = 0$ - ohraničenosť riešenia v nule.

$$u_n(r, \varphi) = R_n(r)\Phi_n(\varphi) = a_n r^{2n-1} \sin((2n-1)\varphi).$$

$$u(r, \varphi) = \sum_{n=1}^{\infty} a_n r^{2n-1} \sin((2n-1)\varphi).$$

Nenulová okrajová podmienka:

$$u(2, \varphi) = 1 = \sum_{n=1}^{\infty} a_n 2^{2n-1} \sin((2n-1)\varphi).$$

Fourierove koeficienty:

$$A_n = a_n 2^{2n-1} = \frac{4}{\pi} \int_0^{\frac{\pi}{2}} \sin((2n-1)\varphi) d\varphi = \frac{4}{\pi(2n-1)}.$$

$$a_n = \frac{4}{\pi(2n-1)2^{2n-1}}.$$

$$Výsledok: u(r, \varphi) = \sum_{n=1}^{\infty} \frac{4}{\pi(2n-1)} \left(\frac{r}{2}\right)^{2n-1} \sin((2n-1)\varphi).$$