(1) Riešte okrajovú úlohu

$$\triangle u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,\ 0 < x < 2,\ 0 < y < 1,$$
$$\frac{\partial u}{\partial y}(x,0) = u(x,1) = u(0,y) = 0,\ \frac{\partial u}{\partial x}(2,y) = 1.$$

$u(x,y) = X(x)Y(y),$
$X''(x) - \lambda X(x) = 0$
$Y''(y) + \lambda Y(y) = 0,\ Y'(0) = Y(1) = 0.$
$\lambda > 0,\ Y(y) = c_1 \cos\sqrt{\lambda}y + c_2 \sin\sqrt{\lambda}y,$
$Y'(y) = -c_1\sqrt{\lambda}\sin\sqrt{\lambda}y + c_2\sqrt{\lambda}\cos\sqrt{\lambda}y,$
$Y'(0) = c_2\sqrt{\lambda} = 0 \Rightarrow c_2 = 0,\ c_1 = 1,$
$Y(y) = \cos\sqrt{\lambda}y,\ Y(1) = \cos\sqrt{\lambda} = 0 \Rightarrow \sqrt{\lambda} = \frac{(2k-1)\pi}{2},$
$\lambda \equiv \lambda_k = \frac{(2k-1)^2\pi^2}{4},\ Y(y) \equiv Y_k(y) = \cos\frac{(2k-1)\pi}{2}y,\ k = 1, 2, ...$

$X_k'' - \lambda_k X_k(x) = 0,\ X_k'' - \frac{(2k-1)^2\pi^2}{4}X_k(x) = 0$
$X_k(x) = a_k \cosh\frac{(2k-1)\pi}{2}x + b_k \sinh\frac{(2k-1)\pi}{2}x.$
$X_k(0) = a_k = 0 \Rightarrow X_k(x) = b_k \sinh\frac{(2k-1)\pi}{2}x.$

$u_k(x,y) = X_k(x)Y_k(y) = b_k \sinh\frac{(2k-1)\pi}{2}x\cos\frac{(2k-1)\pi}{2}y,$
$u(x,y) = \sum_{k=1}^{\infty} u_k(x,y) = \sum_{k=1}^{\infty} b_k \sinh\frac{(2k-1)\pi}{2}x\cos\frac{(2k-1)\pi}{2}y.$
$1 = \frac{\partial u}{\partial y}(2,y) = \sum_{k=1}^{\infty} b_k\frac{(2k-1)\pi}{2}\cosh(2k-1)\pi\cos\frac{(2k-1)\pi}{2}y.$
$B_k = b_k\frac{(2k-1)\pi}{2}\cosh(2k-1)\pi = 2\int_0^1 1\cos\frac{(2k-1)\pi}{2}y\,dy$
$= \frac{4}{(2k-1)\pi}\sin\frac{(2k-1)\pi}{2} = \frac{4(-1)^{k-1}}{(2k-1)\pi}.$
$b_k = \frac{B_k}{\frac{(2k-1)\pi}{2}\cosh(2k-1)\pi} = \frac{8(-1)^{k-1}}{(2k-1)^2\pi^2\cosh(2k-1)\pi}.$

$$\underline{u(x,y) = \sum_{k=1}^{\infty} \frac{8(-1)^{k-1}}{(2k-1)^2\pi^2\cosh(2k-1)\pi}\sinh\frac{(2k-1)\pi}{2}x\cos\frac{(2k-1)\pi}{2}y.}$$

(2) Riešte okrajovú úlohu

$\triangle u(x,y) = 1,\ x^2 + y^2 < 1,\ u(x,y) = 0,$ ak $x^2 + y^2 = 1.$
$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} = 1,\ 0 < r < 1.$
$r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \phi^2} = r^2,\ 0 \le r < 1,\ 0 \le \phi \le 2\pi,\ u(1,\phi) = 0$
$u(r,\phi) = R(r),\ 0 \le r < 1,\ R(1) = 0.$
$r^2R''(r) + rR'(r) = r^2,\ rR'' + R' = r,$
$(rR'(r))' = r,\ R(1) = 0,$
$rR'(r) = \frac{1}{2}r^2 + C,\ R'(r) = \frac{1}{2}r + \frac{C_1}{r},$
$R(r) = \frac{1}{4}r^2 + C_1\ln r + C_2,\ C_1 = 0,$ pretože riešenie je ohraničené.
$R(1) = \frac{1}{4} + C_2 = 0 \Rightarrow C_2 = -\frac{1}{4},$
$R(r) = u(r,\phi) = \frac{1}{4}r^2 - \frac{1}{4} = \frac{1}{4}(r^2 - 1),$

(3) Riešte okrajovú úlohu

$\Delta u(x,y) = 0,\ x^2+y^2<9,\ y>0;$
$u(x,0)=0,\ \frac{\partial u}{\partial \vec{n}} = 2,$ ak $x^2+y^2=9.$

Riešenie:
$\Delta u(r,\varphi) = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial^2 \varphi} = 0,\ u(r,0)=u(r,\pi)=0,\ \frac{\partial u}{\partial r}(3,\varphi)=2.$
$r^2\Delta u(r,\varphi) = r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial^2 \varphi} = 0,\ 0\le r<3,\ 0<\varphi<\pi.$

Dosadiť $u(r,\varphi)=R(r)\Phi(\varphi)$:
$r^2R''\Phi + rR'\Phi + R\Phi'' = 0/\frac{1}{R\Phi},$

$\frac{r^2R''+rR'}{R} = -\frac{\Phi''}{\Phi} = \lambda.$

$r^2R'' + rR' - \lambda R(r) = 0,\ 0\le r<3,$

$\Phi'' + \lambda\Phi = 0,\ \Phi(0)=\Phi(\pi)=0.$

Z prvej okrajovej podmienky: $\lambda>0,\ \Phi(\varphi) = \sin\sqrt{\lambda}\varphi.$
Z druhej okrajovej podmienky:
$\Phi(\pi) = \sin\sqrt{\lambda}\pi = 0,\ \Rightarrow\ \sqrt{\lambda}\pi = n\pi,\ n=1,2,...$
$\lambda\equiv\lambda_n = n^2,\ \Phi\equiv\Phi_n(\varphi) = \sin n\varphi.$

$r^2R_n'' + rR_n' - n^2R_n(r) = 0,\ 0\le r<3,$
$R_n(r) = a_nr^n + b_nr^{-n},\ b_n=0$ - ohraničenosť riešenia v nule.
$u_n(r,\varphi) = R_n(r)\Phi_n(\varphi) = a_nr^n\sin n\varphi.$

$u(r,\varphi) = \sum_{n=1}^{\infty} a_nr^n\sin n\varphi.$

Nenulová okrajová podmienka:
$\frac{\partial u}{\partial r}(3,\varphi) = 2 = \sum_{n=1}^{\infty} a_nn3^{n-1}\sin n\varphi.$

Fourierove koeficienty:
$A_n = a_nn3^{n-1} = \frac{2}{\pi}\int_0^{\pi} 2\sin n\varphi\, d\varphi = \frac{2[1-(-1)^n]}{n\pi}.$
$a_n = \frac{2[1-(-1)^n]}{3^{n-1}n^2\pi}.$

$$u(r,\varphi) = \sum_{n=1}^{\infty}\frac{6[1-(-1)^n]}{n^2}\left(\frac{r}{3}\right)^n\sin n\varphi.$$

(4) Riešte okrajovú úlohu

$\Delta u(x,y) = 0,\ x^2+y^2<4,\ x>0,\ y>0;$
$u(x,0)=0,\ \frac{\partial u}{\partial x}(0,y)=0,\ u=1,$ ak $x^2+y^2=4.$

Riešenie:
$\Delta u(r,\varphi) = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial^2 \varphi} = 0,\ u(r,0)=u(r,\pi)=0,\ \frac{\partial u}{\partial r}(3,\varphi)=2.$
$r^2\Delta u(r,\varphi) = r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial^2 \varphi} = 0,\ 0\le r<2,\ 0<\varphi<\frac{\pi}{2}.$

Dosadiť $u(r,\varphi)=R(r)\Phi(\varphi)$:
$r^2R''\Phi + rR'\Phi + R\Phi'' = 0/\frac{1}{R\Phi},$

$\frac{r^2R''+rR'}{R} = -\frac{\Phi''}{\Phi} = \lambda.$

$r^2R'' + rR' - \lambda R(r) = 0,\ 0 \le r < 2,$

$\Phi'' + \lambda\Phi = 0,\ \Phi(0) = \Phi'(\frac{\pi}{2}) = 0.$

Z prvej okrajovej podmienky: $\lambda > 0,\ \Phi(\varphi) = \sin\sqrt{\lambda}\varphi.$
Z druhej okrajovej podmienky:
$\Phi'(\frac{\pi}{2}) = \sqrt{\lambda}\cos\sqrt{\lambda}\frac{\pi}{2} = 0,\ \Rightarrow\ \sqrt{\lambda}\frac{\pi}{2} = (2n-1)\frac{\pi}{2},\ n = 1, 2, ...$
$\lambda \equiv \lambda_n = (2n-1)^2,\ \Phi \equiv \Phi_n(\varphi) = \sin(2n-1)\varphi.$

$r^2R_n'' + rR_n' - (2n-1)^2R_n(r) = 0,\ 0 \le r < 2,$
$R_n(r) = a_nr^{2n-1} + b_nr^{-(2n-1)},\ b_n = 0$ - ohraničenosť riešenia v nule.
$u_n(r,\varphi) = R_n(r)\Phi_n(\varphi) = a_nr^{2n-1}\sin(2n-1)\varphi.$

$u(r,\varphi) = \sum_{n=1}^{\infty} a_nr^{2n-1}\sin(2n-1)\varphi.$

Nenulová okrajová podmienka:
$u(2,\varphi) = 1 = \sum_{n=1}^{\infty} a_n2^{2n-1}\sin(2n-1)\varphi.$

Fourierove koeficienty:
$A_n = a_n2^{2n-1} = \frac{4}{\pi}\int_0^{\frac{\pi}{2}}\sin(2n-1)\varphi\,d\varphi = \frac{4}{(2n-1)\pi}.$
$a_n = \frac{4}{2^{2n-1}(2n-1)\pi}.$

$$u(r,\varphi) = \sum_{n=1}^{\infty}\frac{4}{(2n-1)\pi}\left(\frac{r}{2}\right)^{2n-1}\sin(2n-1)\varphi.$$

(5) Riešte úlohu na vlastné hodnoty a vlastné funkcie

$\Delta u + \lambda u(x,y) = 0,\ 0 < x < \pi,\ 0 < y < 2,$
$u(0,y) = u(\pi,y) = \frac{\partial u}{\partial y}(x,0) = u(x,2) = 0.$

Riešenie:

$u(x,y) = X(x)Y(y),\ \lambda = \mu + \nu.$
Jednorozmerné úlohy na vlastné hodnoty a vlastné funkcie:

a) $X'' + \mu X(x) = 0,\ X(0) = X(\pi) = 0,$

b) $Y'' + \nu Y(y) = 0,\ Y'(0) = Y(2) = 0.$

a): Z prvej okrajovej podmienky: $\mu > 0,\ X(x) = \sin\sqrt{\mu}x,$

Z druhej okrajovej podmienky:
$X(\pi) = \sin(\sqrt{\mu}\pi) = 0 \Rightarrow \sqrt{\mu}\pi = m\pi.$

$$\mu \equiv \mu_m = m^2,\ X \equiv X_m(x) = \sin mx.$$

b): Z druhej okrajovej podmienky: $\nu > 0$,
Z prvej okrajovej podmienky $Y(y) = \cos\sqrt{\nu}y$,

Z druhej okrajovej podmienky:
$Y(2) = \cos\sqrt{\nu}2 = 0 \Rightarrow \sqrt{\nu}2 = (2n-1)\frac{\pi}{2}$.

$$\nu \equiv \nu_n = \left[\frac{(2n-1)\pi}{4}\right]^2,\ Y \equiv Y_n(y) = \cos\frac{(2n-1)\pi}{4}y.$$

Spolu:

$$\lambda \equiv \lambda_{m,n} = \mu_m + \nu_n = m^2 + \left[\frac{(2n-1)\pi}{4}\right]^2,$$

$$u \equiv u_{m,n}(x,y) = X_m(x)Y_n(y) = \sin mx \cos\frac{(2n-1)\pi}{4}y,\ m,n = 1,2,...$$

(6) Riešte úlohu na vlastné hodnoty a vlastné funkcie

$\Delta u + \lambda u(x,y) = 0,\ 0 < x < 1,\ 0 < y < 2,$
$\frac{\partial u}{\partial x}u(0,y) = u(1,y) = u(x,0) = u(x,2) = 0.$

Riešenie:

$u(x,y) = X(x)Y(y),\ \lambda = \mu + \nu.$
Jednorozmerné úlohy na vlastné hodnoty a vlastné funkcie:

a) $X'' + \mu X(x) = 0,\ X'(0) = X(1) = 0,$

b) $Y'' + \nu Y(y) = 0,\ Y(0) = Y(2) = 0.$

a): Z druhej okrajovej podmienky: $\mu > 0$,
Z prvej okrajovej podmienky: $X(x) = \cos\sqrt{\mu}x$,
Z druhej okrajovej podmienky:
$X(1) = \cos(\sqrt{\mu}) = 0 \Rightarrow \sqrt{\mu} = (2m-1)\frac{\pi}{2}$.

$$\mu \equiv \mu_m = \left[\frac{(2m-1)\pi}{2}\right]^2,\ X \equiv X_m(x) = \cos\frac{(2m-1)\pi}{2}x.$$

b): Z prvej okrajovej podmienky: $\nu > 0,\ Y(y) = \sin\sqrt{\nu}y$,

Z druhej okrajovej podmienky:
$Y(2) = \sin\sqrt{\nu}2 = 0 \Rightarrow \sqrt{\nu}2 = n\pi$.

$$\nu \equiv \nu_n = \left(\frac{n\pi}{2}\right)^2,\ Y \equiv Y_n(y) = \sin\frac{n\pi}{2}y.$$

Spolu:

$$\lambda \equiv \lambda_{m,n} = \mu_m + \nu_n = \left[\frac{(2m-1)\pi}{2}\right]^2 + \left(\frac{n\pi}{2}\right)^2 = \left(\frac{\pi}{2}\right)^2[(2m-1)^2 + n^2],$$

$$u \equiv u_{m,n}(x,y) = X_m(x)Y_n(y) = \cos\frac{(2m-1)\pi}{2}x \sin\frac{n\pi}{2}y,\ m,n = 1,2,...$$