An optimal control problem for an elastic beam in a dynamic contact with a rigid obstacle

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Outline

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Stationary case

Optimal design problem for a unilaterally supported beam:

I. Hlaváček, I. Bock and J. Lovíšek: Optimal control of a variational inequality with applications to structural analysis. I. Optimal design of a beam with a unilateral support. *Appl. Math. Optim.* **11** (1984), 111-143.

I. Bock and J. Lovíšek: Optimal control problems for variational inequalities with controls in coefficients and in unilateral constraints. *Aplikace Matematiky* **32** (1987), 301-314.

State problem:
Stationary (elliptic) variational inequality
with a variable thickness of a beam as a control variable.

Uniquely solved state problem enables a simple formulation of the Optimal control problem.

Dynamic case

The same geometrical situation, another mechanical phenomena.
Nonstationary (hyperbolic) variational inequality as the state problem.

No uniqueness of a solution.
Problems with *a priori* estimates.

Restrictions in formulating the cost functional in the Optimal design problem.

An elastic beam

An elastic beam of the length $L > 0$.
The variable thickness $e(x), \ 0 < e_1 \le e(x) \le e_2, \ x \in [0, L]$.

The material and geometrical characteristic $d > 0$.
The material density $\rho = 1$.
The rigid inner obstacle $\Phi : [0, L] \mapsto \mathbb{R}$.

The beam acting under the perpendicular load
$f : Q \to \mathbb{R}, \ Q = (0, T] \times (0, L)$.

$u_0 : (0, L) \mapsto \mathbb{R}, \ v_0 : (0, L) \mapsto \mathbb{R}$
- the initial displacement and velocity.

The classical formulation

The initial-boundary value problem
with an unknown contact force $g : (0, T] \mapsto \mathbb{R}$

$$e(x)\ddot{u} + d\left(e^3(x)u_{xx}\right)_{xx} = f(t,x) + g(t,x),\ (t,x) \in Q, \tag{1}$$

$$u_{xx}(t,0) = [e^3(x)u_{xx}]_x(t,0) = 0,\ t \in (0,T] \tag{2}$$

$$u_{xx}(t,L) = [e^3(x)u_{xx}]_x(t,L) = 0,\ t \in (0,T] \tag{3}$$

$$u \geq \Phi + \frac{1}{2}e,\ g \geq 0,\ (u - \Phi - \frac{1}{2}e)g = 0,\ (t,x) \in Q \tag{4}$$

$$u(0,x) = u_0(x),\ \dot{u}(0,x) = v_0(x),\ x \in (0,L). \tag{5}$$

Hilbert spaces

Hilbert space $H \equiv L_2(0, L)$,
with the inner product and the norm

$$(y, z) = \int_0^L y(x)z(x)\,dx,\ |y|_0 = (y, y)^{1/2},\ u,\ v \in H$$

Hilbert space $V = H^2(0, L) = \{y \in L_2(0, L) :\ y'' \in L_2(0, L)\}$,
with the inner product and the norm

$$((y, z)) = \int_0^L [y(x)z(x) + y''(x)z''(x)]\,dx,\ \|y\| = ((y, y))^{1/2},\ y,\ z \in V$$

Convex sets

Basic Banach space $\mathcal{V} = L_\infty(I, V)$

Convex sets

$$\mathcal{K}(e) :=$$
$$\{y \in \mathcal{V} : y(t,x) \geq \Phi(x) + \frac{1}{2}e(x) \text{ for a.e.} (t,x) \in (0,T] \times [0,L]\}$$
$$K(e) :=$$
$$\{v \in V : \; v(x) \geq \Phi(x) + \frac{1}{2}e(x) \text{ for all } x \in [0,L]\}$$

Weak solution - variational inequality

Let $f \in L_2(Q)$, $\varPhi \in V$, $u_0 \in K(e)$, $v_0 \in H$;
$e \in H^2[0, L]$, $0 < e_1 \leq e \leq e_2$, $\langle\langle \cdot, \cdot \rangle\rangle$ means the duality between the spaces $\mathcal{V}^*$ and $\mathcal{V}$ as the extension of the inner product in the space $L_2(Q)$.

Definition

Function $u \in \mathcal{K}(e)$ is a weak solution of the problem (1)-(5) if $\ddot{u} \in \mathcal{V}^*$, the initial conditions (5) are fulfilled in a certain generalized sense and there holds the inequality

$$\langle\langle \ddot{u}, e(y - u) \rangle\rangle + d \int_Q e^3(x) u_{xx} (y - u)_{xx} \, dt \, dx \geq$$
$$\int_Q f(x)(y(t, x) - u(t, x)) \, dt \, dx, \ \forall y \in \mathcal{K}(e).$$

Penalized problem

We define for $\varepsilon > 0$
the *penalized problem* of the variational form:

To find $u_\varepsilon \in \mathcal{V}$ such that $\ddot{u}_\varepsilon \in L_2(I; V^*)$ and

$$\langle\langle \ddot{u}_\varepsilon, ey \rangle\rangle + \int_Q \left[de^3(x) u_{\varepsilon_{xx}} y_{xx} - \varepsilon^{-1}\left(u_\varepsilon - \Phi(x) - \frac{1}{2}e(x)\right)^- y \right] dt\, dx$$
$$= \int_Q fy\, dt\, dx \qquad \forall y \in L_2(I, V), \tag{6}$$

$$u_\varepsilon(0, x) = u_0(x),\ \dot{u}_\varepsilon(0, x) = v_0(x),\ x \in (0, L). \tag{7}$$

Galerkin method

The existence of a solution to the penalized problem can be verified by the Galerkin method.

Theorem. *There exists a solution $u \equiv u_\varepsilon$ of the problem* (6), (7) *fulfilling the estimate*

$$\begin{aligned}&\|\dot{u}_\varepsilon\|^2_{L_\infty(I,L_2(0,L))} + \|u_\varepsilon\|^2_{L_\infty(I,V)} \le C(d,e_1,e_2,u_0,v_0,f),\\ &C(d,e_1,e_2,u_0,v_0,f) =\\ &\left(\frac{2}{e_1}+\frac{1}{de_1^3}\right)\left(e_2|v_0|_0^2 + de_2^3\|u_0\|^2 + \frac{2}{e_1}\|f\|^2_{L_1(I,L_2(0,L))}\right).\end{aligned} \tag{8}$$

Crucial estimates

The *a priori* estimates and the convergence process derived in the previous section imply the estimate

$$\|\dot{u}_\varepsilon\|^2_{L_\infty(I,L_2(0,L))} + \|u_\varepsilon\|^2_{L_\infty(I,V)} + \varepsilon^{-1}\|u_\varepsilon^- - \Phi - \frac{1}{2}e\|^2_{C(\bar{I},L_2(0,L))} \leq c_2. \tag{9}$$

Let us set $y(t,x) \equiv 1$ in (6).
The estimate (9) implies after integration through I:

$$0 \leq \varepsilon^{-1} \int_Q (u_\varepsilon - \Phi - \frac{1}{2}e)^- \, dt\, dx \leq c_3, \; \|\ddot{u}_\varepsilon\|_{L_1(I;V^*)} \leq c_4. \tag{10}$$

The limit process

There exists a sequence $\varepsilon_n \searrow 0$, a function $u \in \mathcal{V} \cap H^1(Q)$ and a functional $g \in (L_\infty(Q))^*$ such that $\ddot{u} \in (L_\infty(I, V))^*$, $\dot{u} \in L_\infty(I, H) \cap C_w(\bar{I}, H)$ and for $u_n \equiv u_{\varepsilon_n}$ the following convergences hold

$$\begin{aligned}
\ddot{u}_n &\rightharpoonup^* \ddot{u} && \text{in } (L_\infty(I; V))^* \\
\dot{u}_n &\rightharpoonup \dot{u} && \text{in } L_2(I; V), \\
u_n &\rightharpoonup^* u && \text{in } \mathcal{V}, \\
u_n &\to u && \text{in } C(\bar{I}, H^{2-\delta}(0, L))\ \forall \delta > 0, \\
u_n^- - \Phi - \frac{1}{2} e &\to 0 && \text{in } C(\bar{I}, H), \\
\varepsilon_n^{-1}(u_n^- - \Phi - \frac{1}{2} e) &\rightharpoonup^* g && \text{in } (L_\infty(Q))^*.
\end{aligned} \tag{11}$$

The operator form

Define the operator $A(e) : V \mapsto V^*$ by

$$\langle A(e)u, y\rangle_* = d \int_0^L e^3(x) u_{xx} y_{xx} \, dx, \ u, \ y \in V. \tag{12}$$

The limit function u fulfils the equation in V^*

$$e\ddot{u} + A(e)u = f + g, \tag{13}$$

where $e\ddot{u} \in \mathcal{V}^*$ is defined by $\langle\langle e\ddot{u}, y\rangle\rangle = \langle\langle \ddot{u}, ey\rangle\rangle \ \forall y \in \mathcal{V}$.

Existence theorem

Theorem. *Let* $u_0 \in K(e),\ v_0 \in H,\ f \in L_2(Q),\ \Phi \in V,\ e \in V,$ $0 < e_1 \le e \le e_2$. *Then there exists a weak solution of the State problem* (1)-(5) *fulfilling the estimate*

$$\|\dot{u}\|^2_{L_\infty(I,L_2(0,L))} + \|u\|^2_{L_\infty(I,V)} \le C(d, e_1, e_2, u_0, v_0, f) \qquad (14)$$

with the constant $C(d, e_1, e_2, u_0, v_0, f)$ defined in (8).

Cost functional, admissible controls

Consider a cost functional

$$J : \mathcal{V} \times C^2([0, L]) \mapsto \mathbb{R}^+$$

fulfilling

$$u_n \rightharpoonup^* u \text{ in } \mathcal{V},\ e_n \to e \text{ in } C^2([0, L]) \Rightarrow J(u, e) \le \liminf_{n\to\infty} J(u_n, e_n).$$

The set of admissible thicknesses

$$E_{ad} = \{ e \in H^3(0, L) : \ 0 < e_1 \le e(x) \le e_2 \\ \forall x \in [0, L],\ \|e\|_{H^3(0,L)} \le e_3 \}$$

E_{ad} is compact in $H^2(0, L)$.

Optimal control problem

We introduce the space of functions

$$\mathcal{W} := \{v \in L_\infty(I; L_2(0,L)) : \exists\, \dot{v} \in L_\infty(I; V)^* \text{ and } \{v_n\} \subset H^1(I; L_2(0,L)) \text{ such that } v_n \rightharpoonup^* v \text{ in } L_\infty(I; L_2(0,L)),\ \dot{v}_n \rightharpoonup^* \dot{v} \text{ in } L_\infty(I; V)^*\}.$$

Optimal control problem $\mathcal{P}$.

To find a couple (u_*, e_*) such that

$$J(u_*, e_*) \le J(u, e)\ \forall (u, e) \in U_{ad}(e) \times E_{ad}, \tag{15}$$

$$U_{ad}(e) = \{u \in \mathcal{K}(e) : \ \dot{u} \in \mathcal{W},\ u \text{ is a weak solution of } (1)-(5), \|\dot{u}\|^2_{L_\infty(I;L_2(0,L))} + \|u\|^2_{L_\infty(I;V)} \le C_1\}.$$

$U_{ad}(e) \neq \emptyset$ for every $e \in E_{ad}$.

Existence of the optimal thickness

Let $\{(u_n, e_n)\} \in U_{ad}(e_n) \times E_{ad}$ be a minimizing sequence i.e.

$$\lim_{n\to\infty} J(u_n, e_n) = \inf_{U_{ad}(e)\times E_{ad}} J(u, e).$$

There exists $(u_*, e_*) \in \mathcal{K}(e_*) \times E_{ad}$ and a subsequence (u_n, e_n) such that

$$e_n \to e_* \text{ in } H^2(0, L), \; u_n \rightharpoonup u_* \text{ in } \mathcal{V}.$$

It can be verified that (u_*, e_*) is a solution of the minimization problem (15).

Theorem.

There exists a solution of the Optimal control problem $\mathcal{P}$.

Thank you