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SOME REMARKS ON OPERATOR GENERALIZED EFFECT ALGEBRAS

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Introduction and preliminaries

The present contribution is based on the paper [5] which is a joint work with Z. Riečanová.

Effect algebras and generalized effect algebras are examples of quantum structures the theory of which is developing in recent years. The first example of an effect algebra including noncompatible and unsharp elements was given by Foulis and Bennett in [1]. It is the set of Hilbert space effects $\mathcal{E}(\mathcal{H}) = \{A \in \mathcal{B}(\mathcal{H}) \mid 0 \le A \le I\}$, i.e. the set of positive bounded operators such that the operator $I - A$ is positive (here I is the identity operator). (Let us mention that $\mathcal{E}(\mathcal{H})$ is important in the mathematical description of unsharp measurement in Hilbert space quantum mechanics.) The set $\mathcal{E}(\mathcal{H})$ is endowed by a partial operation $\oplus$ such that for $A, B \in \mathcal{E}(\mathcal{H})$, $A \oplus B$ is defined and is equal to $A + B$ iff $A + B \le I$. ($A + B \le I$ means that $I - A - B$ is a positive operator.) Then $\mathcal{E}(\mathcal{H})$ with operation $\oplus$ satisfies the following

Definition 1 (Foulis and Bennett, [1])**.** A partial algebra $(E; \oplus, 0, 1)$ is called an *effect algebra* if 0,1 are two distinct elements and $\oplus$ is a partially defined binary operation on E which satisfy the following conditions for any $x, y, z \in E$:

(E1) $x \oplus y = y \oplus x$ if $x \oplus y$ is defined,

(E2) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ if one side is defined,

(E3) for every $x \in E$ there exists a unique $y \in E$ such that $x \oplus y = 1$ (we put $x' = y$),

(E4) If $1 \oplus x$ is defined then $x = 0$.

Generalizations of effect algebras (without a top element 1) have been studied, they are called generalized effect algebras.

Definition 2. (1) A *generalized effect algebra* $(E;\, \oplus,\, 0)$ is a set E with an element $0 \in E$ and partial binary operation $\oplus$ satisfying for any $x, y, z \in E$ conditions

(GE1) $x \oplus y = y \oplus x$ if one side is defined,

(GE2) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ if one side is defined,

(GE3) If $x \oplus y = x \oplus z$ then $y = z$,

(GE4) If $x \oplus y = 0$ then $x = y = 0$,

(GE5) $x \oplus 0 = x$ for all $x \in E$.

(2) $Q \subseteq E$ is called a *sub-generalized effect algebra* of E if and only if out of elements $x, y, z \in E$ with $x \oplus y = z$ at least two are in Q then $x, y, z \in Q$.

Every sub-generalized effect algebra of E is a generalized effect algebra in its own rigt and every effect algebra is a generalized effect algebra in a natural sense (with the same operation $\oplus$).

In [1] the effect algebra $\mathcal{E}(\mathcal{H})$ was modelled by (bounded) operators in Hilbert space. Later, effect algebras and generalized effect algebras were modelled by objects of different kind (e.g. fuzzy sets) or they were regarded as abstract structures. The paper [5] (on which

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the present contribution is based) is an attempt to show some more examples of generalized effect algebras modelled by (possible unbounded) operators in Hilbert space. They are first such examples after the classical paper [1]. Up to now, more works concerning this question appeared, e.g. [2], [3], [4], [6].

Results

Let $\mathcal{H}$ be a complex Hilbert space. Let us recall that a linear operator A on $\mathcal{H}$ is densely defined if for its domain $D(A) \subseteq \mathcal{H}$ it holds $\overline{D(A)} = \mathcal{H}$. A is positive if $(Ax, x) \geq 0$ for all $x \in D(A)$. A is bounded if there exists such $K \in \mathbb{R}$ that $||Ax|| \leq K||x||$ for all $x \in D(A)$. A is unbounded if it is not bounded.

Theorem 3 ([5]). *Let $\mathcal{H}$ be a complex Hilbert space and let $D \subseteq \mathcal{H}$ be a linear subspace dense in $\mathcal{H}$ (i.e., $\bar{D} = \mathcal{H}$). Let*

$$\mathcal{G}_D(\mathcal{H}) = \{A : D \to \mathcal{H} \mid A \textit{ is a positive linear operator defined on } D\}.$$

Then $(\mathcal{G}_D(\mathcal{H}); \oplus, 0)$ is a generalized effect algebra where 0 is the null operator and $\oplus$ is the usual sum of operators defined on D. In this case $\oplus$ is a total operation.

So here D is a fixed dense linear subspace in $\mathcal{H}$ and all positive operators defined on D form a generalized effect algebra ($\oplus$ being the usual operator sum).

The operators $A \in \mathcal{G}_D(\mathcal{H})$ may be unbounded. So $(\mathcal{G}_D(\mathcal{H}); \oplus, 0)$ is an example of a generalized effect algebra modelled by (possibly unbounded) operators on Hilbert space $\mathcal{H}$.

The next Theorem is about bounded operators.

Theorem 4 ([5]). *Let $\mathcal{H}$ be a complex Hilbert space and $D \subseteq \mathcal{H}$ be a dense linear subspace of $\mathcal{H}$. Then the set of all bounded positive linear operators on D form a sub-generalized effect algebra of the generalized effect algebra $\mathcal{G}_D(\mathcal{H})$ of all positive linear operators on D with respect to usual addition of operators, which in this case is a total operation.*

So bounded operators form a generalized effect algebra. Of course the common domain D can be taken to be the whole space $\mathcal{H}$.

Now we show a generalized effect algebra including all unbounded positive linear operators densely defined on $\mathcal{H}$, without fixed domain $D \subseteq \mathcal{H}$.

If A, B are linear operators with domains $D(A), D(B), D(A) \subseteq D(B)$ then $B|_{D(A)}$ is the restriction of B to $D(A)$. In the following, $A + B$ means the usual addition of operators.

Theorem 5 ([5]). *Let $\mathcal{H}$ be an infinite dimensional complex Hilbert space. Let*

$$\begin{aligned} \mathcal{V}(\mathcal{H}) = \{A : \mathcal{H} \to \mathcal{H} \mid & A \textit{ is a positive linear operator with the domain} \\ D(A) = \mathcal{H} \textit{ if } A \textit{ is bounded and with } & \overline{D(A)} = \mathcal{H} \textit{ if } A \textit{ is unbounded}\} \end{aligned}$$

Let $\oplus$ be a partial binary operation on $\mathcal{H}$ defined by $A \oplus B = A + B$ with $D(A \oplus B) = \mathcal{H}$ for any bounded $A, B \in \mathcal{V}(\mathcal{H})$ and $A \oplus B = B \oplus A = A + B|_{D(A)}$ with $D(A \oplus B) = D(A)$ if A is unbounded and B is bounded.

Then $(\mathcal{V}(\mathcal{H}); \oplus, 0)$ is a generalized effect algebra. Moreover,

$$\mathcal{B}(\mathcal{H}) = \{A : \mathcal{H} \to \mathcal{H} \mid A \textit{ is a bounded positive linear operator on } \mathcal{H}\}$$

is a sub-generalized effect algebra of $\mathcal{V}(\mathcal{H})$ with respect to inherited $\oplus$-operation, which becomes total on $\mathcal{B}(\mathcal{H})$.

So we take all positive unbounded densely defined linear operators and all bounded positive linear operators defined on whole $\mathcal{H}$. We add either two bounded operators or a bounded operator to an unbounded operator. We don't add two unbounded operators. (In some later works, however, one also adds two unbounded operators if they have the same domain.)

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