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Cyclic quasianalytic contractions

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Let $\mathcal{H}$ be an infinite dimensional, separable, complex Hilbert space; $\mathcal{L}(\mathcal{H})$ denotes the set of bounded, linear operators acting on $\mathcal{H}$. Let $T \in \mathcal{L}(\mathcal{H})$ be a contraction: $\|T\| \leq 1$. We assume that T is absolutely continuous, that is T is the orthogonal sum of a completely non-unitary contraction and an absolutely continuous unitary operator. We recall that the pair (X, V) is a unitary asymptote of T, if (i) V is an absolutely continuous unitary operator acting on a Hilbert space $\mathcal{K}$, (ii) $X \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ is a contractive mapping intertwining T with $V : \|X\| \leq 1, XT = VX$, and (iii) for any similar contractive intertwining pair (X', V') there exists a unique contractive transformation $Y \in \mathcal{L}(\mathcal{K}, \mathcal{K}')$ such that $YV = V'Y$ and $X' = YX$. For the existence and uniqueness of unitary asymptotes we refer to [BK] (see also [K1]). Let us assume also that T is of class C_{10}, which means that T is asymptotically non-vanishing: $\lim_{n\to\infty} \|T^n x\| > 0$ for every $0 \neq x \in \mathcal{H}$, and the adjoint T^* is stable: $\lim_{n\to\infty} \|T^n x\| = 0$ for every $x \in \mathcal{H}$. Then the intertwining mapping X is injective. The quasianalytic spectral set $\pi(T)$ of T can be introduced in terms of the spectral subspaces of V. Let E denote the spectral measure of V, and for any measurable subset ω of the unit circle $\mathbb{T}$ let $\mathcal{K}(\omega) = E(\omega)\mathcal{K}$ be the corresponding spectral subspace. For any $h \in \mathcal{H}$, ω_h denotes the smallest measurable set on $\mathbb{T}$ such that $Xh \in \mathcal{K}(\omega_h)$. Then $\pi(T)$ is the largest measurable set on $\mathbb{T}$ satisfying the condition $\pi(T) \subset \omega_h$ for every non-zero $h \in \mathcal{H}$. The residual set $\omega(T)$ of T is the measurable support of E, that is the complement of the largest measurable set Ω on $\mathbb{T}$ such that $E(\Omega) = 0$. The sets $\pi(T)$ and $\omega(T)$ are determined up to sets of zero Lebesgue measure. The contraction T is quasianalytic, if $\pi(T) = \omega(T)$. For further details see [K2].

In [K4] we introduced and studied some distinctive classes of quasianalytic contractions. We recall that $\mathcal{L}_0(\mathcal{H})$ consists of the operators $T \in \mathcal{L}(\mathcal{H})$ satisfying the conditions: (i) T is a C_{10}-contraction, (ii) T is quasianalytic, and (iii) the unitary operator V is cyclic. The subclass $\mathcal{L}_1(\mathcal{H})$ consists of those operators $T \in \mathcal{L}_0(\mathcal{H})$, which satisfy also the additional condition: (iv) $\pi(T) = \mathbb{T}$. Note that every operator $T \in \mathcal{L}_1(\mathcal{H})$ has a rich invariant subspace lattice Lat T; see [K3]. Let us consider also the class $\widetilde{\mathcal{L}}(\mathcal{H})$ of those absolutely continuous contractions $T \in \mathcal{L}(\mathcal{H})$, which are non-stable (i.e., $\lim_{n\to\infty} \|T^n x\| > 0$ for some $x \in \mathcal{H}$), and where the unitary asymptote V is cyclic. Clearly $\mathcal{L}_1(\mathcal{H}) \subset \mathcal{L}_0(\mathcal{H}) \subset \widetilde{\mathcal{L}}(\mathcal{H})$.

For an operator $T \in \mathcal{L}(\mathcal{H})$, $\{T\}' = \{C \in \mathcal{L}(\mathcal{H}) : CT = TC\}$ denotes the commutant of T, and Hlat $T = \mathrm{Lat}\{T\}'$ stands for the hyperinvariant subspace lattice of T. The Invariant Subspace Problem (ISP) asks whether every operator $T \in \mathcal{L}(\mathcal{H})$ has a non-trivial invariant subspace, that is Lat $T \neq \{\{0\}, \mathcal{H}\}$. The Hyperinvariant Subspace Problem (HSP) asks whether every operator $T \in \mathcal{L}(\mathcal{H}) \setminus \mathbb{C}I$ has a non-trivial hyperinvariant subspace. These problems are arguably the most challenging open questions in operator theory. We know from [K2] that the (HSP) in the class $\widetilde{\mathcal{L}}(\mathcal{H})$ is equivalent to the (HSP) in the class $\mathcal{L}_0(\mathcal{H})$. Furthermore, if the (HSP) has positive answer in $\widetilde{\mathcal{L}}(\mathcal{H})$, then the (ISP) has an affirmative answer in the class of contractions T, where T or T^* is non-stable. As we mentioned earlier, the (ISP) in $\mathcal{L}_1(\mathcal{H})$ is answered affirmatively. Actually, a lot of information is at our disposal on the structure of operators in $\mathcal{L}_1(\mathcal{H})$, which may be helpful in the study of the (HSP) in this class; see [K3]. It was proved in [K4] that if $T \in \mathcal{L}_0(\mathcal{H})$ and $\pi(T)$ contains an arc then there exists $T_1 \in \mathcal{L}_1(\mathcal{H})$ such that $\{T\}' = \{T_1\}'$, and so Hlat T = Hlat T_1. We are able to show now that the whole class $\mathcal{L}_0(\mathcal{H})$ is strongly related to $\mathcal{L}_1(\mathcal{H})$, proving the following theorem.

Theorem 1. *For every operator $T \in \mathcal{L}_0(\mathcal{H})$ there exists $T_1 \in \mathcal{L}_1(\mathcal{H})$ commuting with T : $TT_1 = T_1T$.*

Since the commutants $\{T\}'$ and $\{T_1\}'$ are abelian, the equation $T_1T = TT_1$ implies $\{T\}' = \{T_1\}'$, and so Hlat $T =$ Hlat T_1. Thus we obtain the following corollary.

Corollary 2. *The (HSP) in the class $\mathcal{L}_0(\mathcal{H})$ is equivalent to the (HSP) in the class $\mathcal{L}_1(\mathcal{H})$.*

This result is related to those in [FPN], [FP], [BFP] and [K3].

The operator T_1 in $\mathcal{L}_1(\mathcal{H}) \cap \{T\}'$ is provided as a function of T, using the Sz.-Nagy–Foias functional calculus; see [NFBK, Chapter III]. The spectral mapping theorem established in [K4] is applied. The existence of a function $f \in H^\infty$, satisfying the conditions $f(T) \in \mathcal{L}_0(\mathcal{H})$ and $\pi(f(T)) = f(\pi(T)) = \mathbb{T}$, is based on Theorem 3 below.

Let Λ denote the linear measure on $\mathbb{C}$, coinciding with the Lebesgue measure on $\mathbb{T}$ and $\mathbb{R}$. A domain $G \subset \mathbb{C}$ is called a *circuler comb domain* if it is obtained from the open unit disc $\mathbb{D}$ by deleting countably many radial segments of the form $\{r\zeta : a < r < 1\}$ with some $0 < a < 1$ and $\zeta \in \mathbb{T}$.

Theorem 3. *If E is a measurable subset of the unit circle $\mathbb{T}$ of positive (linear) measure, then there are a compact set $K \subset E$ and a conformal map f from $\mathbb{D}$ onto a circular comb domain such that f can be extended to a continuous function on the closed unit disc $\mathbb{D}^-$, $f(K) = \mathbb{T}$, and $\Lambda(f(H)) = 0$ for every Borel subset H of K of zero measure.*

The proof of Theorem 3 is based on application of potential theoretic tools.

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