8th Workshop Functional Analysis and its Applications, September 5–10, 2011, Nemecká, Slovakia, 12–16

Method of Reliable Solution in Homogenization

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Abstract. Method of reliable solution solves the problem of equations with uncertain data. It looks for solutions and data of the equation giving maximum of a functional describing dangerous situations – the worst scenario method. The contribution formulates the reliable solution problem in homogenization of elliptic equations and surveys the results obtained by Luděk Nechvátal and the author.

Keywords: homogenization, reliable solution, worst scenario method.

MSC 2010: 35B27, 35J60.

Introduction

Partial differential equations are used for modeling of real problems in engineering practice. The model contains material constants in constitute relations which are not known exactly, since they are obtained by measurements and are loaded by errors. Thus only intervals, where values of these constants can occur, are known. The equations are said to have "uncertain" input data. The uncertain coefficients cause uncertain solutions.

Deterministic approach to this problem called "Worst Scenario Method" was proposed by I. Babuška and further developed by I. Hlaváček and J. Chleboun, see [3]. It is based on the following idea: A set $\mathscr{U}^{\mathrm{ad}}$ of admissible data a is defined, it is often described by a Cartesian product of the intervals and the corresponding set of problems $(\mathbf{P}[a])$, $a \in \mathscr{U}^{\mathrm{ad}}$ is considered. According to the character of the real engineering problem a special cost functional $\Phi(a)$ is chosen. It measures dangerous situations like peaks of deformation or stress, extreme temperature or extreme temperature slope. Then the maximum of Φ on $\mathscr{U}^{\mathrm{ad}}$ is looked for, i. e. we look for the worst case that can happen on the admissible data. The method is called "Reliable Solution Method" or "Worst Scenario Method". In general, the maximum of the problem need not exist. In linear problems the maximum lies on the boundary of $\mathscr{U}^{\mathrm{ad}}$, in the case of nonlinear equation it need not be true.

Homogenization is a mathematical method for modeling materials with periodic structure, especially composite materials. To solve these problems numerically is impossible since fine structure of the material requires even finer triangulation which leads to enormous number of equations. Thus for computation reasons the material with fine structure is replaced by an equivalent homogeneous material; in mathematical settings the equation with highly oscillating coefficients is approximated by a constant coefficient equation giving almost the same solutions.

Approach proposed by I. Babuška is based on construction of a sequence of materials with finer and finer structure; in mathematical setting a sequence of equations having coefficients with diminishing period is studied. The method, see e. g. [1], enables to compute the so-called homogenized coefficients of the so-called homogenized equation from knowledge of the coefficient in the period, i. e. from the parameters of the components of the composite and their space distribution. The main results of the homogenization theory is to find formulae for the homogenized coefficients and to prove convergence of the corresponding solutions.

The aim of the contribution is to describe the method of reliable solution in homogenization and to outline the results obtained by L. Nechvátal and the author. The case of linear elliptic equation is formulated, generalization to nonlinear case is only sketched.

[1]This research is supported by Brno University of Technology, Specific Research project no. FSI-S-11-3.

1 Preliminaries

A sequence $E=\{\varepsilon_n\}_{n=1}^{\infty}$ of small positive numbers ε_n tending to zero is called the *scale*. In homogenization instead of a subscript $n\in\mathbb{Z}$ sequences are denoted with a superscript ε_n from the scale E, but the n in ε_n is omitted.

Let Y be a basic cell, usually it is the unit cube in $\mathbb{R}^N$, i. e. $Y=\langle 0,1)^N$. A collection of shifted cells $Y_k=Y+k=\{y+k \mid y\in Y\}$ with $k=(k_1,\dots,k_n)$, $k_i\in\mathbb{Z}$ is a pavement of the space $\mathbb{R}^N$, i. e. the cells Y_k are disjoint and their union over $k\in\mathbb{Z}^N$ covers the whole $\mathbb{R}^N$. A function $a(y)$ defined on $\mathbb{R}^N$ is called to be Y-periodic, if $a(y+k)=a(y)$ holds for each $y\in\mathbb{R}^N$ and each $k\in\mathbb{Z}^N$. The ε-scaled cell Y will be denoted by Y^ε and k-shifted ε-scaled cell by Y_k^ε, i. e. $Y_k^\varepsilon=\{\varepsilon(y+k)\mid y\in Y\}$.

Let Ω be a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary Γ. Let a be a Y-periodic function. Then relation

$$a^\varepsilon(x)=a\left(\frac{x}{\varepsilon}\right)\equiv a\left(\frac{x_1}{\varepsilon},\dots,\frac{x_N}{\varepsilon}\right), \qquad x\in\Omega \tag{1}$$

defines a sequence $\{a^\varepsilon \mid \varepsilon\in E\}$ of Y^ε-periodic functions on Ω with diminishing period ε. We shall use standard notation of function spaces: $L^p(\Omega)$ is the Lebesgue space of functions integrable in the p-th power (bounded measurable functions in case of $p=\infty$), $W^{1,2}(\Omega)$ is the Sobolev space of functions with square integrable first derivatives and $W_0^{1,2}(\Omega)$ its subspace with zero traces on Γ. Space of Y-periodic functions will be denoted by the subscript *per*, e. g. $W^{1,p}_{\text{per}}(Y)$.

2 Linear elliptic problem

For sake of simplicity we start with a linear differential equation with coefficient matrix a

$$-\operatorname{div}(a\,\nabla u_a)\equiv-\sum_{i=1,j}^{N}\frac{\partial}{\partial x_i}\left(a_{ij}(x)\,\frac{\partial u}{\partial x_j}\right)=f \qquad \text{in}\quad \Omega$$

completed with the homogeneous Dirichlet boundary condition

$$u_a=0 \qquad \text{on}\quad \partial\Omega.$$

The solution is taken in the so-called weak sense, i. e.

Problem $(\mathbf{P}[a])$ *Find a function* $u_a\in W_0^{1,2}(\Omega)$ *satisfying*

$$\mathcal{a}_a(u_a,v)\equiv\int_\Omega\sum_{i,j=1}^{N}a_{ij}(x)\frac{\partial u_a}{\partial x_j}\,\frac{\partial v}{\partial x_i}\,\mathrm{d}x=\int_\Omega fv\,\mathrm{d}x. \qquad \textit{for all}\ \ v\in W_0^{1,2}(\Omega).$$

Throughout the paper we shall assume that $f\in L^2(\Omega)$ and the coefficients a_{ij} are bounded measurable functions forming a symmetric positive definitive matrix, i. e.

$$a_{ij}\in L^\infty(\Omega), \qquad a_{ji}=a_{ij}, \qquad \alpha\sum_{i=1}^{N}\xi_i^2\le\sum_{i,j=1}^{N}a_{ij}(x)\xi_j\xi_i\le M\sum_{i=1}^{N}\xi_i^2 \quad \forall\,\xi\in\mathbb{R}^N. \tag{2}$$

The class of all such coefficient matrix functions a satisfying (2) with constants $0<\alpha\le M$ will be denoted by $\mathscr{E}(\alpha,M)$.

Following the well known Lax-Milgram lemma Problem $(\mathbf{P}[a])$ for $a\in\mathscr{E}(\alpha,M)$ admits unique solution u_a and it, in addition, satisfies estimate

$$\|u_a\|_{1,2}\le\frac{1}{\alpha}\,\|f\|_2\ . \tag{3}$$

3 Formulation of the homogenization problem and its solution

For any $\varepsilon \in E$ and a Y-periodic matrix function $a : \Omega \to \mathbb{R}^{N\times N}$ satisfying (2) by (1) we obtain a ε-periodic functions a^{ε}_{ij} and then the corresponding problem with ε-periodic coefficients:

Problem ($\mathbf{P}[a^{\varepsilon}]$) *Find a function* $u_{a^{\varepsilon}} \in W^{1,2}_0(\Omega)$ *satisfying*

$$\mathcal{a}_{a^{\varepsilon}}(u_{a^{\varepsilon}}, v) \equiv \int_{\Omega} \sum_{i,j=1}^{N} a_{ij}\left(\frac{x}{\varepsilon}\right) \frac{\partial u_{a^{\varepsilon}}}{\partial x_j} \frac{\partial v}{\partial x_i} \,\mathrm{d}x = \int_{\Omega} f\, v \,\mathrm{d}x \qquad \textit{for all} \;\; v \in W^{1,2}_0(\Omega). \tag{4}$$

Following the previous section the problem ($\mathbf{P}[a^{\varepsilon}]$) admits unique solution $u_{a^{\varepsilon}}$. Taking a scale E of diminishing parameters ε we obtain a sequence $\{u_{a^{\varepsilon}}\}$ of corresponding solutions. Due to (3) the sequence is bounded in $W^{1,2}(\Omega)$.

The well known result, see e. g. [1], says that the sequence of solutions $u_{a^{\varepsilon}}$ converge in $W^{1,2}(\Omega)$ weakly to a function u_{b^a} which is a solution to the same type problem but with the so-called homogenized coefficients – matrix of constant function b^a:

Problem ($\mathbf{P}[b^{a}]$) *Find a function* $u_{b^a} \in W^{1,2}_0(\Omega)$ *satisfying*

$$\mathcal{a}_{b^a}(u_{b^a}, v) \equiv \int_{\Omega} \sum_{i,j=1}^{N} b^a_{ij} \frac{\partial u_{b^a}}{\partial x_j} \frac{\partial v}{\partial x_i} \,\mathrm{d}x = \int_{\Omega} f v \,\mathrm{d}x. \qquad \textit{for all} \;\; v \in W^{1,2}_0(\Omega).$$

The homogenized coefficients b^a can be computed from the matrix function $a(y)$ on Y:

$$b^a_{ij} = \int_Y \left[a_{ij}(y) + \sum_{k=1}^{N} a_{ik}(y) \frac{\partial w^k_a}{\partial y_j}(y) \right] \mathrm{d}y, \tag{5}$$

where the auxiliary functions w^k_a are Y-periodic solutions to the so-called cell-problem

Problem ($\mathbf{P_{per}}[a]$) *Find a vector function* $w_a = (w^1_a, \ldots, w^N_a)$, $w^k_a \in W^{1,2}_{per}(Y)$ *satisfying*

$$\int_Y \left[\sum_{i,j=1}^{N} a_{ij}(y) \frac{\partial w^k_a}{\partial y_j} \frac{\partial \varphi}{\partial y_i} + \sum_{i=1}^{N} a_{ik}(y) \frac{\partial \varphi}{\partial y_i} \right] \mathrm{d}y = 0 \quad \forall \varphi \in W^{1,2}_{per}(Y) \;\; \textit{and} \quad \int_Y w^k_a(y) \,\mathrm{d}y = 0. \tag{6}$$

It can be proved that the homogenized coefficients b^a_{ij} form also a positive definitive matrix. If a_{ij} are symmetric, then the matrix b^a is in the class satisfying estimates (2) with the same constants, i. e. if $a \in \mathscr{E}(\alpha, M)$ then also $b^a \in \mathscr{E}(\alpha, M)$.

4 Method of reliable solution

Let $\mathscr{U}^{\mathrm{ad}}$ be a subset of $\mathscr{E}(\alpha, M)$ and $f \in L^2(\Omega)$. Then for each $a \in \mathscr{U}^{\mathrm{ad}}$ we have a homogenization problem consisting of a sequence $\varepsilon \in E$ of problems ($\mathbf{P}[a^{\varepsilon}]$), sequence of the solutions $u_{a^{\varepsilon}}$, homogenized coefficient matrix b^a, the homogenized problem ($\mathbf{P}[b^a]$) and its homogenized solution u_{b^a}. Let us choose a functional Φ on $\mathscr{U}^{\mathrm{ad}}$ which evaluates hazardousness of the situation, i. e. of the homogenized solution u_{b^a}.

We have to prove that the functional Φ is bounded and attains its maximum on $\mathscr{U}^{\mathrm{ad}}$. Then we evaluate the maximum and find the data which yields this maximum.

The proof is based on the following well known property. If the set $\mathscr{U}^{\mathrm{ad}}$ is compact and $\Phi : \mathscr{U}^{\mathrm{ad}} \to \mathbb{R}$ is continuous on $\mathscr{U}^{\mathrm{ad}}$ with respect to the same topology, then the image of $\Phi(\mathscr{U}^{\mathrm{ad}})$ in $\mathbb{R}$ is also compact, i. e. it is a bounded closed set in $\mathbb{R}$, which has a maximum.

The set of admissible data $\mathscr{U}^{\mathrm{ad}}$. Homogenization theory was built for modeling of composite materials. For simplicity we shall deal with a two component composite material with periodic structure. Let Y_1 be a closed subset of the cell Y representing reinforcing fibres and its complement $Y_0 = Y - Y_1$ representing the matrix. Let the parameters p^1_{ij} of the fibres (e. g. stiffness or conductivity, etc.) be in closed intervals I^1_{ij} and parameters p^0_{ij} of the matrix be in closed intervals I^0_{ij}. Then the coefficients a_{ij} will be defined as a piecewise constant functions

$$a_{ij}(y) = \begin{cases} p^1_{ij} & \text{for } y \in Y_1, \\ p^0_{ij} & \text{for } y \in Y_0 \end{cases} \tag{7}$$

on the cell Y and by periodicity extended to $\mathbb{R}^N$.

The set of all such functions $a_{ij}(y)$ with $p^1_{ij} \in I^1_{ij}$ and $p^0_{ij} \in I^0_{ij}$ will be the set of admissible functions $\mathscr{U}^{\mathrm{ad}}$. By its construction it is a bounded closed subset in $L^\infty_{\mathrm{per}}(Y)$ and due to its finite dimension it is compact – it can be represented by a product of finite number of bounded closed intervals. Since the parameters describes real materials we can suppose that $\mathscr{U}^{\mathrm{ad}}$ is a subset of $\mathscr{E}(\alpha, M)$ for some convenient $0 < \alpha \le M$.

Generalization to finite number of components of the composite material causes no problem: the cell Y is decompose into more subsets $Y = Y_0 \cup Y_1 \cup \cdots \cup Y_k$, compactness of $\mathscr{U}^{\mathrm{ad}}$ remains.

Criterion functional. How to choose the functional Φ evaluating dangerous situations? Since in dimension $N \ge 2$ functions from $W^{1,2}(\Omega)$ need not be continuous, we cannot take values of the solution in a single point. Instead of it we choose a small subset Ω^* of Ω of positive measure which covers the critical place and put the integral mean of over it. In homogenization we test the values of the homogenized solution u_{b^a}, thus

$$\Phi(a) = \frac{1}{|\Omega^*|} \int_{\Omega^*} u_{b^a}(x) \, \mathrm{d}x,$$

Another possibility is to test gradient of the homogenized solution u_{b^a}. The functional Φ can depend even on the values of a, in homogenization we can take into account also e. g. correctors. Looking for a maximum, the functional may be only upper semi-continuous only.

Let us introduce the result proved in [4]:

THEOREM. *The functional Φ on $\mathscr{U}^{\mathrm{ad}}$ attains its maximum.*

Idea of the proof. Having a compact set $\mathscr{U}^{\mathrm{ad}}$ the proof of existence of the maximum is based on continuity of the functional Φ, i. e. the property

$$a_n \to a_0 \qquad \Longrightarrow \qquad \Phi(a_n) \to \Phi(a_0).$$

Indeed, let m^* be the supremum of Φ on $\mathscr{U}^{\mathrm{ad}}$, in general, it can be also plus infinity. Let us consider a sequence $\{a_n\}$ in $\mathscr{U}^{\mathrm{ad}}$ such that $\Phi(a_n)$ tends to m^*. Since $\mathscr{U}^{\mathrm{ad}}$ is compact, the sequence $\{a_n\}$ contains a subsequence $\{a_{n'}\}$ and there exists an element $a^* \in \mathscr{U}^{\mathrm{ad}}$ such that $a_{n'} \to a^*$. Due to continuity $\Phi(a^*) = \lim_{n' \to \infty} \Phi(a_{n'}) = m^*$. Since m^* is a value of Φ on an element of $\mathscr{U}^{\mathrm{ad}}$, m^* is finite and Φ attains its maximum on $\mathscr{U}^{\mathrm{ad}}$ in a^*.

In the model problem (4) on $\mathscr{U}^{\mathrm{ad}}$ the topology of the uniform convergence can be chosen, i. e. the maximizing sequence $\{a_n\}$ is uniformly converging. Let $a, a' \in \mathscr{U}^{\mathrm{ad}}$, then using notation of Section 3 the continuity of the functional Φ is a consequence of the following estimates:

$$\begin{aligned} |\Phi(a) - \Phi(a')| &\le \text{const.}\, \|u_{b^a} - u_{b^{a'}}\|_{W^{1,2}(\Omega)}, \\ \|u_{b^a} - u_{b^{a'}}\|_{W^{1,2}(\Omega)} &\le \text{const.}\, \max_{i,j} \left| b^a_{ij} - b^{a'}_{ij} \right|, \\ \max_{i,j} \left| b^a_{ij} - b^{a'}_{ij} \right| &\le \text{const.}\, \|w_a - w_{a'}\|_{W^{1,2}_{\mathrm{per}}(Y, \mathbb{R}^N)}, \\ \|w_a - w_{a'}\|_{W^{1,2}_{\mathrm{per}}(Y)} &\le \text{const.}\, \|a - a'\|_{L^\infty(Y, \mathbb{R}^{N \times N})}. \end{aligned} \tag{8}$$

5 Generalization to nonlinear monotone operator problem

There are several generalizations of the linear problem. Let us introduce the case of an equation with monotone operator published in [2]:

PROBLEM ($\mathbf{P}_M[a]$) *Find a function* $u_a \in W_0^{1,2}(\Omega)$ *satisfying*

$$\int_\Omega \sum_{i=1}^N a_i(x, \nabla u_a) \frac{\partial v}{\partial x_i} \, \mathrm{d}x = \int_\Omega f v \, \mathrm{d}x \qquad \text{for all } \; v \in W_0^{1,2}(\Omega).$$

To obtain solvability of the problems, the coefficients $a_i(y,\xi) : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ are supposed to satisfy strong monotonicity and Lipschitz continuity in ξ with some constants $0 \le \alpha \le M$

$$\begin{aligned} \textstyle\sum_{i=1}^N (a_i(y,\xi) - a_i(y,\eta))(\xi_i - \eta_i) &\ge \alpha \cdot \textstyle\sum_{i=1}^N (\xi_i - \eta_i)^2 && \forall y,\xi,\eta \in \mathbb{R}^N, \\ |a_i(y,\xi) - a_i(y,\eta)| &\le M \cdot |\xi - \eta| && \forall y,\xi,\eta \in \mathbb{R}^N . \end{aligned} \tag{9}$$

Since the homogenization problem is studied, the coefficients are supposed to be Y-periodic in y. To obtain a compact set, each function $a_i(y,\xi)$ in $\mathscr{U}^{\mathrm{ad}}$ is piecewise constant in y on each part Y_j of the cell Y like in (7), i. e. $a_i(y,\xi) = p_i^j(\xi)$ for $y \in Y_j$. Further each $p_i^j(\xi)$ depends on ξ_i only. Outside of a fixed bounded interval $J_i = \langle \xi_i^l, \xi_i^r \rangle$ the function $p_i^j(xi_i)$ is extended to $\mathbb{R}$ linearly with a given slope c_i^j. Due to (9) inside the interval J_i the functions $p_i^j(\xi_i)$ satisfy assumptions of the Arzelà-Ascoli theorem and thus the functions $a_i(y,\xi)$ form a set $\mathscr{U}_i^{\mathrm{ad}}$ compact with respect to uniform convergence. Then cartesian product $\mathscr{U}_1^{\mathrm{ad}} \times \cdots \times \mathscr{U}_N^{\mathrm{ad}}$ yields a compact set $\mathscr{U}^{\mathrm{ad}}$ of admissible data, for details see [2].

As in the linear case for a scale E by (1) we obtained a sequence of problems ($\mathbf{P}_M[a_\varepsilon]$) which yields a sequence of solutions u_{a^ε}. In theory of monotone operator homogenization there exist results giving formulae for homogenized coefficients analogous to (5), (6). Using estimates analogous to (8) continuity of Φ and existence of a maximum over $\mathscr{U}^{\mathrm{ad}}$ can be proved.

Conclusion

In the contribution the method of reliable solution in periodic homogenization was introduced for linear problems and a generalization to nonlinear monotone operator problem from [2] was outlined. Choosing a convenient compact set of admissible data $\mathscr{U}^{\mathrm{ad}}$ and suitable estimates analogous to (8) in other generalization including evolutionary problems the existence of solution to the worst scenario method can be proved.

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