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On the commutant of asymptotically non-vanishing contractions

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Joint work with László Kérchy

One of the main methods of examining non-normal operators, acting on Hilbert spaces, is the theory of contractions. This area of operator theory was developed by Béla Sz.-Nagy and Ciprian Foias from the dilation theorem of Sz.-Nagy. Sz.-Nagy and Foias classified the contractions according to their asymptotic behaviour. They got strong structural results in the case when the contraction and its adjoint are simultaneously asymptotically non-vanishing. However, basic questions are still open (e.g. the hyperinvariant and the invariant subspace problems), when only the contraction is asymptotically non-vanishing. In this case one can associate a unitary asymptote to the contraction on a canonical way. The connection with this unitary operator is manifested in an algebra-homomorphism between the commutants of these operators. This commutant mapping is among the few links which relate the contraction to a well-understood operator. It can be exploited to get structure theorems or stability results; see e.g. [Ba], [K3], [K5], [K6], [KL] and [KV]. Hence it is of interest to study its properties. Our purpose was to examine the injectivity of the commutant mapping. One of the results states that this mapping can be injective even in the case when the contraction has a non-trivial stable subspace. Various characterizations of injectivity are provided.

Now we give the basic definitions and fix the notation. Let $\mathcal{H}$ and $\mathcal{K}$ be non-zero, complex, separable Hilbert spaces, and let $\mathcal{L}(\mathcal{H},\mathcal{K})$ stand for the system of bounded, linear transformations from $\mathcal{H}$ to $\mathcal{K}$. Then $\mathcal{L}(\mathcal{H}) := \mathcal{L}(\mathcal{H},\mathcal{H})$ is the set of operators acting on $\mathcal{H}$. The *commutant* $\{T\}'$ of $T \in \mathcal{L}(\mathcal{H})$ consists of those operators $C \in \mathcal{L}(\mathcal{H})$, which commute with T: $TC = CT$. For any operators $A \in \mathcal{L}(\mathcal{H})$ and $B \in \mathcal{L}(\mathcal{K})$, the intertwining set $\mathcal{I}(A,B)$ consists of those transformations $Y \in \mathcal{L}(\mathcal{H},\mathcal{K})$, which satisfy the condition $YA = BY$. Any invertible transformation $Y \in \mathcal{L}(\mathcal{H},\mathcal{K})$ is said to be an affinity. If Y is injective (i.e. $\ker Y = \{0\}$) and has dense range (i.e. $(\operatorname{ran} Y)^- = \mathcal{K}$), then it is called a quasiaffinity.

Let $T \in \mathcal{L}(\mathcal{H})$ be a contraction: $\|T\| \leq 1$. The vector $h \in \mathcal{H}$ is asymptotically vanishing or stable for T if $\lim_{n\to\infty} \|T^n h\| = 0$. The subspace $\mathcal{H}_0 = \mathcal{H}_0(T)$ of all stable vectors is called the *stable subspace* of T. It is clearly hyperinvariant for T, that is invariant for every operator which commutes with T. We recall that T is of class $C_{0\cdot}$ if T is stable, that is $\mathcal{H}_0 = \mathcal{H}$. If $\mathcal{H}_0 = \{0\}$, then T is of class $C_{1\cdot}$ and is called asymptotically strongly non-vanishing. T is asymptotically non-vanishing or of class $C_{*\cdot}$ if $\mathcal{H}_0 \neq \mathcal{H}$. We say that T is of class $C_{\cdot j}$ ($j \in \{0,*,1\}$) if its adjoint is of class $C_{j\cdot}$. Finally the class C_{ij} consists of those operators, which are both in $C_{i\cdot}$ and $C_{\cdot j}$ ($i,j \in \{0,*,1\}$). We refer to [SzNF] in connection with the theory of contractions.

Unitary asymptotes of operators were studied in several papers, see e.g. [Bea, Chapter XII], [K1], [K2] and [K4]. Here we recall the definition given in [BK]. The pair (X,W) is a contractive unitary intertwining pair of the contraction $T \in \mathcal{L}(\mathcal{H})$ if $W \in \mathcal{L}(\mathcal{K})$ is unitary and $X \in \mathcal{I}(T,W)$ is contractive. We say that (X,W) is a *unitary asymptote* of T, if for any other contractive unitary intertwining pair (Y,U) there exists a unique $Z \in \mathcal{I}(W,U)$ such that $Y = ZX$ and $\|Z\| \leq 1$. Such a unitary asymptote (X,W) always exists and is unique up to isomorphism. Given any $C \in \{T\}'$, the pair (XC,W) is a unitary intertwining pair for T. By the universality of (X,W), there exists exactly one $D \in \{W\}'$ such that $XC = DX$. Thus we obtain a mapping

$$\gamma = \gamma_T \colon \{T\}' \to \{W\}', \quad C \mapsto D,$$

Key words: commutant mapping, unitary asymptote, contraction.

which turns out to be a contractive algebra-homomorphism. This γ is called the *commutant mapping* of the contraction T.

Some of our main results are the following:

Theorem 1. **(B. Sz-Nagy and C. Foias)** *Every contraction $T \in \mathcal{L}(\mathcal{H})$ has the matrix form*

$$\begin{bmatrix} T_{00} & T_{01} \\ 0 & T_{11} \end{bmatrix},$$

where the matrix is meant in the decomposition $\mathcal{H}_0 \oplus \mathcal{H}_1$, $T_{00} \in C_{0\cdot}(\mathcal{H}_0)$ and $T_{11} \in C_{1\cdot}(\mathcal{H}_1)$.

Theorem 2. *If the commutant mapping γ of the contraction $T \in \mathcal{L}(\mathcal{H})$ is injective, then*
(i) $\mathcal{I}(T_{11}, T_{00}) = \{0\}$,
(ii) $\overline{\sigma_{\rm ap}(T_{00}^*)} \cap \sigma_{\rm ap}(T_{11}) \neq \emptyset$,
(iii) $\sigma_p(T) \cap \overline{\sigma_p(T^*)} \cap \mathbb{D} = \emptyset$, *and*
(iv) *there is no such direct decomposition $\mathcal{H} = \mathcal{M}_0 \dot{+} \mathcal{M}_1$, where $\mathcal{M}_0, \mathcal{M}_1$ are invariant subspaces of T and $\{0\} \neq \mathcal{M}_0 \subset \mathcal{H}_0$.*

Theorem 3. *Let us assume that the stable component T_{00} of the contraction T satisfies the following conditions*:
(i) $\sigma_p(T_{00}^*) \subset \overline{\sigma_p(T_{00})}$,
(ii) T_{00}^* *has a generating root subspace system.*
Then γ is injective if and only if $\sigma_p(T) \cap \overline{\sigma_p(T^*)} \cap \mathbb{D} = \emptyset$.

Theorem 4. *Let us assume that the contractions $T \in (\mathcal{H})$ and $T' \in (\mathcal{H}')$ are quasisimilar. Then γ_T is injective if and only if $\gamma_{T'}$ is injective.*

Theorem 5. *There exists a contraction $T \in (\mathcal{H})$ such that the conditions* (i)–(iv) *of* Theorem 2 *hold, but the commutant mapping γ of T is not injective.*

Details are communicated in [GK].

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