

Príklad 1  $x' = y - x^2 - x$   
 $y' = 3x - y - x^2$

Prvé body:

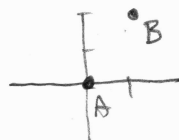
$$\begin{array}{r} y - x^2 - x = 0 \\ 3x - y - x^2 = 0 \\ \hline 2x - 2x^2 = 0 \end{array}$$

$$\begin{array}{ll} x_1 = 0 & x_2 = 1 \\ y_1 = 0 & y_2 = 2 \end{array}$$

Systém má 2 prvé body

$$A = [0, 0]$$

$$B = [1, 2]$$



Linearizácia

$$\begin{array}{ll} \frac{\partial f}{\partial x} = -2x - 1 & \frac{\partial f}{\partial y} = 1 \\ \frac{\partial g}{\partial x} = 3 - 2x & \frac{\partial g}{\partial y} = -1 \end{array}$$

V okoli bodu A  
 $A = [0, 0]$

$$DF(A) = \begin{pmatrix} -1 & 1 \\ 3 & -1 \end{pmatrix}$$

$$\begin{vmatrix} -1-\lambda & 1 \\ 3 & -1-\lambda \end{vmatrix} = \lambda^2 + 2\lambda - 2 = 0$$

$$\lambda_1 = \frac{-2 + \sqrt{4+8}}{2} = -1 + \sqrt{3} > 0$$

$$\lambda_2 = -1 - \sqrt{3} < 0$$

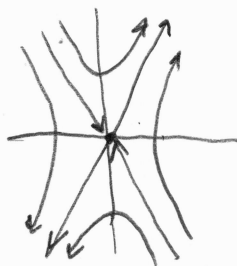
Bod A je sedlo

$$\lambda_1 = -1 + \sqrt{3} \quad \begin{pmatrix} -\sqrt{3} & 1 \\ 3 & -\sqrt{3} \end{pmatrix}$$

$$\vec{v}_1 = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$$

$$\lambda_2 = -1 - \sqrt{3} \quad \begin{pmatrix} \sqrt{3} & 1 \\ 3 & \sqrt{3} \end{pmatrix}$$

$$\vec{v}_2 = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}$$



V okolí bodu  $B = [1, 2]$ 

$$DF(B) = \begin{pmatrix} -3 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\begin{vmatrix} -3-\lambda & 1 \\ 1 & -1-\lambda \end{vmatrix} = \lambda^2 + 4\lambda + 2 = 0$$

Vlastní čísla  $\lambda_1 = \frac{-4 + \sqrt{16-8}}{2} = -2 + \sqrt{2} < 0$

$$\lambda_2 = -2 - \sqrt{2} < 0$$

B - stoka

Vl. Vektory

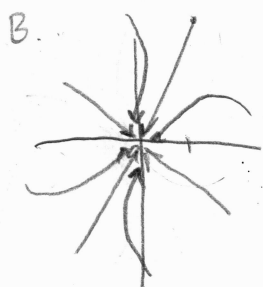
$$\begin{pmatrix} -1-\sqrt{2} & 1 \\ 1 & 1-\sqrt{2} \end{pmatrix}$$

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 1+\sqrt{2} \end{pmatrix}$$

$$\begin{pmatrix} -1+\sqrt{2} & 1 \\ 1 & 1+\sqrt{2} \end{pmatrix}$$

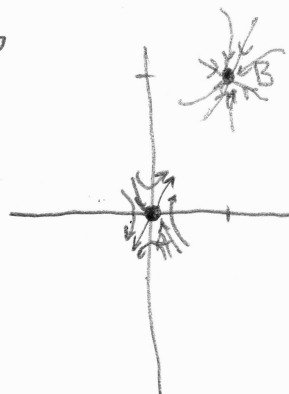
$$\vec{v}_2 = \begin{pmatrix} 1 \\ 1-\sqrt{2} \end{pmatrix}$$

$$y_1(t) = \begin{pmatrix} 1 \\ 1+\sqrt{2} \end{pmatrix} e^{(-2+\sqrt{2})t}$$

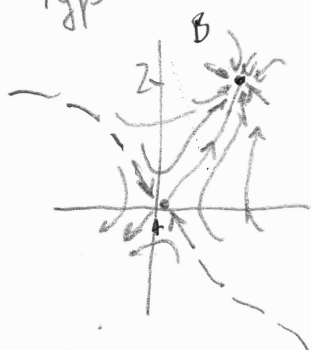


Súhrn

Lokálny FP



Typ



# Veta o linearizácii (Hartman - Grobman)

Nelineárny systém  $\dot{x} = F(x)$  je v okolí hyperbolického pevného bodu  $\bar{x}$  lokálne kvalitatívne ekvivalentný s linearizáciou  $\dot{w} = DF(\bar{x})w$  v okolí (jej pevného bodu)  $\vec{0}$ .

$F$  - hladké

hyperbolický p. bod - vl. čísla  $\lambda_i$  matice  $DF(\bar{x})$  majú  $\operatorname{Re} \lambda_i \neq 0$ .

Príklad 2. 
$$\begin{aligned} x' &= y^2 - 3x + 2 \\ y' &= x^2 - y^2 \end{aligned}$$

Pevné body: 
$$\begin{aligned} y^2 - 3x + 2 &= 0 \\ x^2 - y^2 &= 0 \\ \hline x^2 - 3x + 2 &= 0 \end{aligned}$$

$$x_1 = 1 \quad x_2 = 2$$

$$x^2 = y^2$$

$$\begin{aligned} A &= [1, 1] & C &= [2, 2] \\ B &= [1, -1] & D &= [2, -2] \end{aligned}$$

Linearizácia: 
$$\begin{aligned} \frac{\partial f}{\partial x} &= -3 & \frac{\partial f}{\partial y} &= 2y \\ \frac{\partial g}{\partial x} &= 2x & \frac{\partial g}{\partial y} &= -2y \end{aligned}$$

$$DF = \begin{pmatrix} -3 & 2y \\ 2x & -2y \end{pmatrix}$$

V bode  $A = [1, 1]$  
$$DF(A) = \begin{pmatrix} -3 & 2 \\ 2 & -2 \end{pmatrix} \quad |M - \lambda I| = \begin{vmatrix} 3 - \lambda & 2 \\ 2 & -2 - \lambda \end{vmatrix} = \lambda^2 + 5\lambda + 2 = 0$$

$M = DF(A)$   $\operatorname{Det} M = 2$   $\operatorname{Tr} M = -5$

$$\lambda_{1,2} = \frac{-5 \pm \sqrt{17}}{2}$$

A Stoka

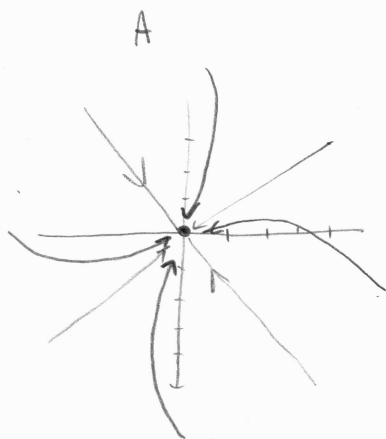
$$\lambda_1: \begin{pmatrix} \frac{1 - \sqrt{17}}{2} & 2 \\ 2 & -\frac{1 - \sqrt{17}}{2} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \vec{0}$$

$$V_1 = \begin{pmatrix} 4 \\ \sqrt{17} - 1 \end{pmatrix}$$

$$\varphi_1(t) = \begin{pmatrix} 4 \\ \sqrt{17} - 1 \end{pmatrix} e^{\frac{-5 + \sqrt{17}}{2} t}$$

$$\lambda_2:$$

$$\varphi_2(t) = \begin{pmatrix} 4 \\ -\sqrt{17} - 1 \end{pmatrix} e^{\frac{-5 - \sqrt{17}}{2} t}$$



V bode  $B = [1, -1]$

$$DF(B) = \begin{pmatrix} -3 & -2 \\ 2 & 2 \end{pmatrix}$$

$$\det M = -2$$

$$|M - \lambda I| = \begin{vmatrix} -3-\lambda & -2 \\ 2 & 2-\lambda \end{vmatrix} = \lambda^2 + \lambda - 2 = 0$$

$$\lambda_{1,2} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

B-sedlo

$$\lambda_1: \begin{pmatrix} -4 & -2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \vec{0}$$

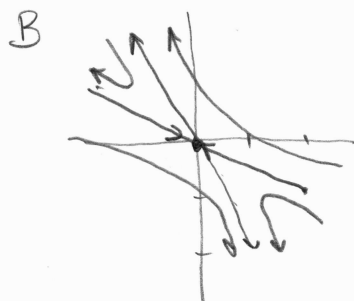
$$\text{vl. vektor } \vec{v}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\varphi_1(t) = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^t$$

$$\lambda_2: \begin{pmatrix} -1 & -2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \vec{0}$$

$$\vec{v}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$\varphi_2(t) = \begin{pmatrix} 2 \\ -1 \end{pmatrix} e^{-2t}$$



V bode  $C = [2, 2]$

$$DF(C) = \begin{pmatrix} -3 & 4 \\ 4 & -4 \end{pmatrix}$$

$$|M - \lambda I| = \begin{vmatrix} -3-\lambda & 4 \\ 4 & -4-\lambda \end{vmatrix} = \lambda^2 + 7\lambda - 4$$

$$\lambda_{1,2} = \frac{-7 \pm \sqrt{65}}{2}$$

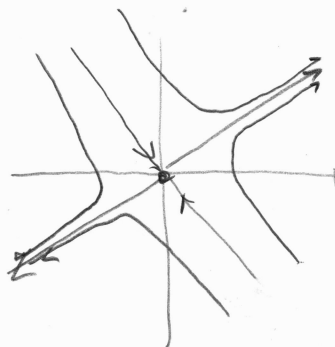
C-sedlo

$$\begin{pmatrix} \frac{1}{2} - \frac{\sqrt{65}}{2} & 4 \\ 4 & -\frac{1}{2} - \frac{\sqrt{65}}{2} \end{pmatrix}$$

$$\vec{v}_1 = \begin{pmatrix} 8 \\ \sqrt{65} - 1 \end{pmatrix}$$

$$\varphi_1(t) = \begin{pmatrix} 8 \\ \sqrt{65} - 1 \end{pmatrix} e^{\frac{-7+\sqrt{65}}{2}t}$$

$$\varphi_2(t) = \begin{pmatrix} 8 \\ -\sqrt{65} - 1 \end{pmatrix} e^{\frac{-7-\sqrt{65}}{2}t}$$



V bode  $D = [2, -2]$

$$DF(D) = \begin{pmatrix} -3 & -4 \\ +4 & +4 \end{pmatrix}$$

$$|M - \lambda I| = \begin{vmatrix} -3 - \lambda & -4 \\ +4 & +4 - \lambda \end{vmatrix} = \lambda^2 - \lambda + 4 = 0$$

$$\lambda_{1,2} = \frac{1 \pm i\sqrt{15}}{2}$$

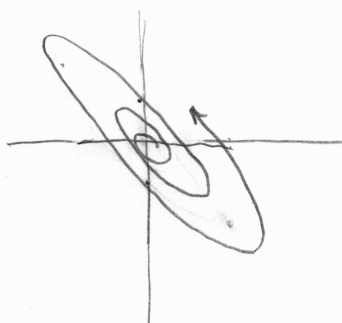
Nest. ohnisko

$$\begin{pmatrix} -\frac{7}{2} - \frac{i\sqrt{15}}{2} & -4 \\ 4 & \frac{7}{2} - \frac{i\sqrt{15}}{2} \end{pmatrix} \begin{pmatrix} 8 \\ -7 - i\sqrt{15} \end{pmatrix}$$

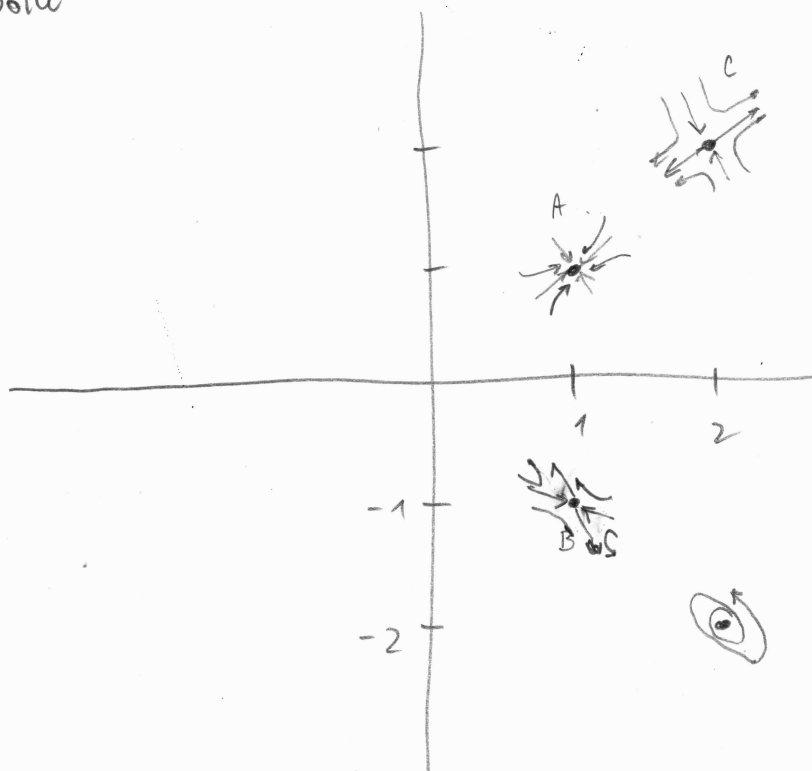
$$\frac{1}{2} + \frac{i\sqrt{15}}{2} t$$

$$\varphi_1(t) = e^{\frac{1}{2}t} \begin{pmatrix} 8 \cos \frac{\sqrt{15}}{2} t \\ -7 \cos \frac{\sqrt{15}}{2} t + \sqrt{15} \sin \frac{\sqrt{15}}{2} t \end{pmatrix}$$

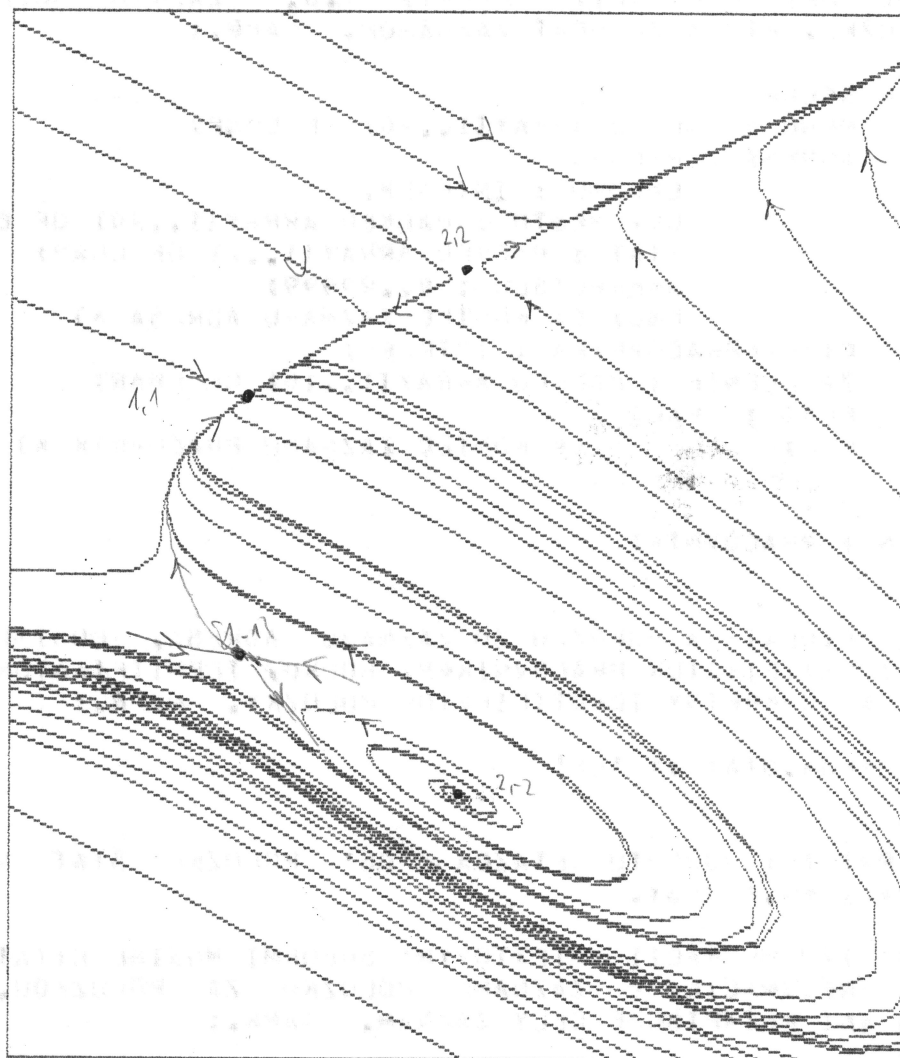
$$\varphi_2(t) = e^{\frac{1}{2}t} \begin{pmatrix} 8 \sin \frac{\sqrt{15}}{2} t \\ -7 \sin \frac{\sqrt{15}}{2} t - \sqrt{15} \cos \frac{\sqrt{15}}{2} t \end{pmatrix}$$



Spolu







$$x' = y^2 - 3x + 2$$

$$y' = x^2 - y^2$$

4. perva body

Príklad.  $\dot{x} = -\mu x + xy$   
 $\dot{y} = 1 - y - x^2$

Riešte pre  $\mu > 0$ . Rozlíšte typické prípady a nájdite bifurkačné hodnoty parametra  $\mu$ .

Príklad. Súťažiaci druhhy  
 $x(t)$  - početný stav druhu A  
 $y(t)$  - početný stav druhu B

Rovnice

$$\begin{aligned} x' &= ax - bx^2 - sxy \\ y' &= cy - dy^2 - rxy \end{aligned} \quad a, b, c, d, r, s > 0$$

Možné scenáre

- I oba druhy vymrú
- II A prežije      III B prežije
- IV oba prežijú

Pevné body

$$\begin{aligned} ax - bx^2 - sxy &= 0 & x(a - bx - sy) &= 0 \\ cy - dy^2 - rxy &= 0 & y(c - dy - rx) &= 0 \end{aligned}$$

I  $x=0, y=0$   $[0, 0]$

II  $y=0, x=\frac{a}{b}$   $[\frac{a}{b}, 0]$

III  $x=0, y=\frac{c}{d}$   $[0, \frac{c}{d}]$

IV  $[\frac{cs-ad}{rs-bd}, \frac{ar-bc}{rs-bd}]$

Nutná podmienka koexistencie

- (i)  $cs - ad > 0, ar - bc > 0, rs - bd > 0$
- (ii)  $cs - ad < 0, ar - bc < 0, rs - bd < 0$

Linearitácia

$$DF(x,y) = \begin{pmatrix} a-2bx-sy & -sx \\ -ry & c-2dy-rx \end{pmatrix}$$

$$\text{v bode } [0,0] : \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$$

$$\lambda_1 = a > 0$$

$$\lambda_2 = c > 0$$

žriedlo

$$\text{vlastné vektory } \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$



$$\text{v bode } \left[\frac{a}{b}, 0\right] : \begin{pmatrix} -a & -s\frac{a}{b} \\ 0 & c-r\frac{a}{b} \end{pmatrix}$$

$$\lambda_1 = -a < 0$$

$$\lambda_2 = \frac{cb-ra}{b}$$

V prípade (i)  $\lambda_2 = 0$   
(ii)  $\lambda_2 > 0$

stoka v prípade (i) sedlo v prípade (ii)

$$\text{vlastné vektory } \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} \frac{sa}{b} \\ \frac{ra-cb-ab}{b} \end{pmatrix}$$

(i)



(ii)



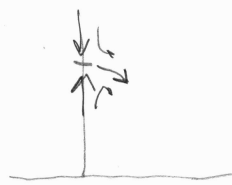
$$\text{v bode } \left[0, \frac{c}{d}\right] : \begin{pmatrix} a-s\frac{c}{d} & 0 \\ -r\frac{c}{d} & -c \end{pmatrix}$$

$$\lambda_1 = \frac{ad-cs}{d}$$

$$\lambda_2 = -c < 0$$

V prípade (i)  $\lambda_2 < 0$   
(ii)  $\lambda_2 > 0$

(i)



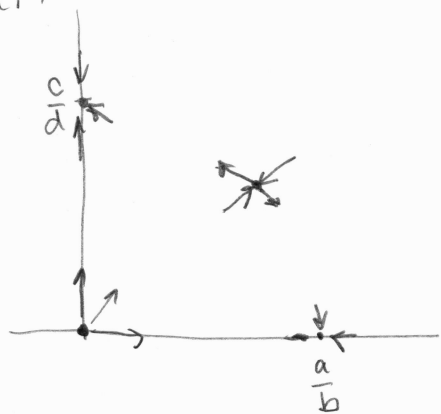
$$\text{v bode } \left[\frac{cs-ad}{rs-bd}, \frac{ar-bc}{rs-bd}\right] :$$

$$\begin{pmatrix} a-2b\frac{cs-ad}{rs-bd} - s\frac{ar-bc}{rs-bd} & -s\frac{cs-ad}{rs-bd} \\ -r\frac{ar-bc}{rs-bd} & c-2d\frac{ar-bc}{rs-bd} - r\frac{cs-ad}{rs-bd} \end{pmatrix} = \frac{1}{rs-bd} \begin{pmatrix} -b(cs-ad) & -s(cs-ad) \\ -r(ar-bc) & -d(ar-bc) \end{pmatrix}$$

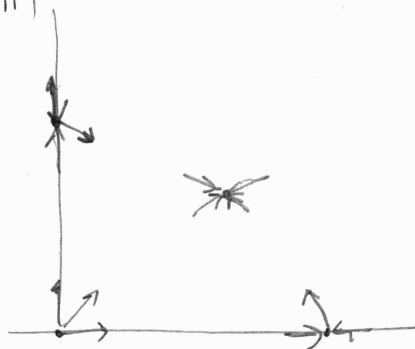
$$\text{Det } M = \frac{-1}{(rs-bd)^2} (cs-ad)(ar-bc)(bd-sr) = \begin{cases} > 0 \text{ v prípade (ii)} & \frac{\text{Tr } M}{2} < 0 \\ < 0 \text{ v prípade (i)} & \text{sedlo} \end{cases}$$

# Lokálne fázové portréty

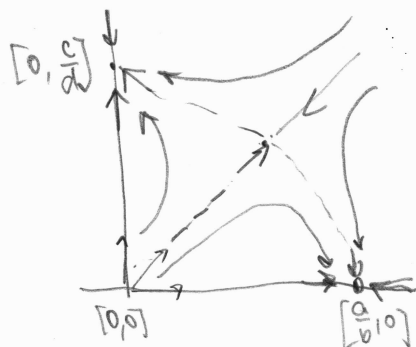
(i)



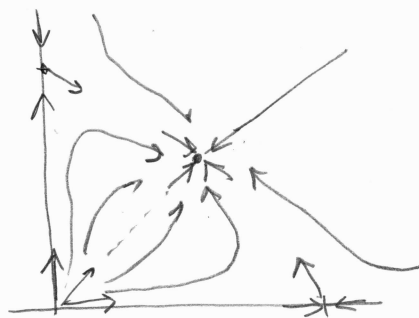
(ii)



## Globálny portrét



Eliminácia



Koexistencia