Laplaceova rovnica.

Uvažujme verziu difúznej alebo vlnovej rovnice v dvoch priestorových premenných x, y

$$u_t = c(u_{xx}+u_{yy}) = c\Delta u$$
$$u_{tt} = c^2(u_{xx}+u_{yy}) = c\Delta u$$

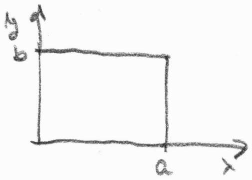
Predpokladajme, že nastal ustálený stav - riešenie sa nemení v čase t

$$0 = \Delta u$$

Riešenie tejto rovnice závisí od tvaru oblasti, na ktorej o nej uvažujeme

Laplaceova rovnica na obdĺžniku.

$$u_{xx}+u_{yy} = 0$$

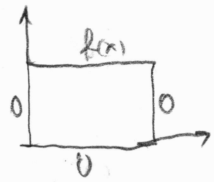


$\langle 0,a\rangle \times \langle 0,b\rangle$

Okrajové podmienky sú podmienky na hranici oblasti.

Najjednoduchšia možnosť

$$u(x,0) = 0 \qquad u(x,b) = f(x)$$
$$u(0,y) = 0 \qquad u(a,y) = 0$$



Riešenie $\quad u(x,y) = X(x)\cdot Y(y)$

$$X''Y + XY'' = 0$$
$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

$$X'' + \lambda X = 0 \qquad Y'' - \lambda Y = 0$$
$$X(0) = 0 \qquad X(a) = 0 \qquad Y(0) = 0$$

$$X(x) = A\cos\sqrt{\lambda}\,x + B\sin\sqrt{\lambda}\,x$$

$$X(0) = 0 \implies A = 0 \qquad X(a) = 0 \implies \sqrt{\lambda}\,a = k\pi \qquad \lambda_k = \left(\frac{k\pi}{a}\right)^2$$

$$X_k(x) = B_k \sin\frac{k\pi}{a}x$$

$$Y'' - \left(\frac{k\pi}{a}\right)^2 Y = 0$$

$$Y_k(y) = C_k e^{-\frac{k\pi}{a}y} + D_k e^{\frac{k\pi}{a}y} = \tilde{C}_k \sinh\frac{k\pi}{a}y + \tilde{D}_k \cosh\frac{k\pi}{a}y$$

$$\cosh z = \frac{e^z + e^{-z}}{2} \qquad \sinh z = \frac{e^z - e^{-z}}{2}$$

$$c_1 e^{-dy} + d_1 e^{+dy} = \frac{c_1+d_1}{2}\cosh dy + \frac{d_1 - c_1}{2}\sinh dy$$

$$c_1 = \tilde{c}_1 - \tilde{d}_1 \qquad d_1 = \tilde{c}_1 + \tilde{d}_1$$

Podmienka $\quad Y(0) = 0 \implies \tilde{D}_k = 0$

$$u(x,y) = \sum_{k=1}^{\infty} b_k \sin\frac{k\pi}{a}x \sinh\frac{k\pi}{a}y$$

$$u(x,b) = \sum_{k=1}^{\infty} b_k \sin\frac{k\pi}{a}x \sinh\frac{k\pi}{a}b = f(x)$$

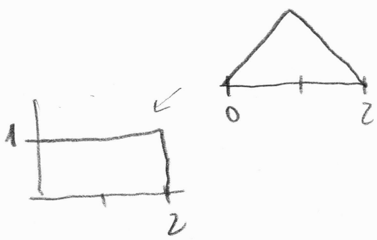
$$b_k \sinh\frac{k\pi}{a}b = \frac{2}{a}\int_0^a f(x)\sin\frac{k\pi}{a}x\,dx$$

$$b_k = \frac{2}{a\sinh\frac{k\pi}{a}b}\int_0^a f(x)\sin\frac{k\pi}{a}x\,dx$$

Príklad. $\quad \Delta u = 0$

$$u(x,0) = 0 \qquad u(x,1) = 1 - |x-1|$$

$$u(0,y) = 0 \qquad u(2,y) = 0$$



Riešenie $\quad u(x,y) = \sum_{k=1}^{\infty} b_k \sin\frac{k\pi}{2}x \sinh\frac{k\pi}{2}y$

$$b_k = \frac{2}{2\sinh\frac{k\pi}{2}}\int_0^2 (1 - |x-1|)\sin\frac{k\pi}{2}x\,dx =$$

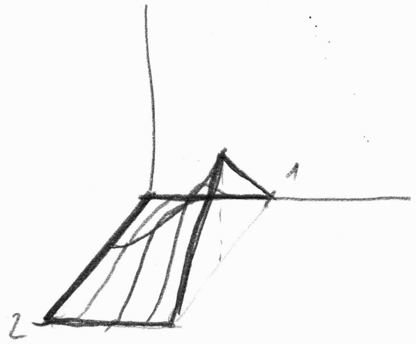
$$= \frac{1}{\sinh\frac{k\pi}{2}}\left(\int_0^1 x\sin\frac{k\pi}{2}x\,dx + \int_1^2 (2-x)\sin\frac{k\pi}{2}x\,dx = \right.$$

$$b_k = \frac{1}{\sinh\frac{k\pi}{2}}\left(\left[x\cdot\frac{-\cos\frac{k\pi}{2}x}{\frac{k\pi}{2}}\right]_0^1 + \int_0^1 \frac{\cos\frac{k\pi}{2}x}{\frac{k\pi}{2}}\,dx + \left[(2-x)\frac{-\cos\frac{k\pi}{2}x}{\frac{k\pi}{2}}\right]_1^2 - \int_1^2 \frac{\cos\frac{k\pi}{2}x}{\frac{k\pi}{2}}\,dx\right) =$$

$$= \frac{1}{\sinh\frac{k\pi}{2}}\left(\frac{-\cos\frac{k\pi}{2}}{\frac{k\pi}{2}} + \left[\frac{\sin\frac{k\pi}{2}x}{\left(\frac{k\pi}{2}\right)^2}\right]_0^1 + \frac{\cos\frac{k\pi}{2}}{\frac{k\pi}{2}} - \left[\frac{\sin\frac{k\pi}{2}x}{\left(\frac{k\pi}{2}\right)^2}\right]_1^2\right) =$$

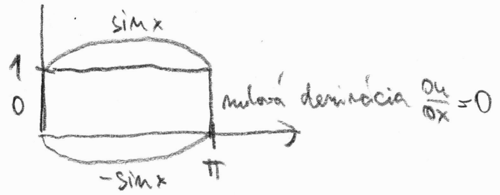
$$= \frac{1}{\sinh\frac{k\pi}{2}}\quad\frac{\sin\frac{k\pi}{2}}{\left(\frac{k\pi}{2}\right)^2}\cdot 2$$

$$u(x,y) = \sum_{k=1}^{\infty} \frac{8}{k^2\pi^2}\,\frac{\sin\frac{k\pi}{2}}{\sinh\frac{k\pi}{2}}\cdot \sin\frac{k\pi}{2}x\cdot \sinh\frac{k\pi}{2}y$$



Problém. Riešte úlohu $\Delta w = 0$ na oblasti $\langle 0,\pi\rangle \times \langle 0,1\rangle$

$$w(x,0) = -\sin x \qquad u\,(x,1) = \sin x$$
$$w(0,y) = 0 \qquad u_x(\pi,y) = 0$$



Laplaceova rovnica na kruhovej oblasti

Uvažujme rovnicu $\Delta u = 0$ na oblasti $x^2+y^2 \le r_0^2$

Oblasť popíšeme pomocou polárnych súradníc

$$x = r\cos\varphi$$
$$y = r\sin\varphi \qquad 0 \le r \le r_0 \quad 0 \le \varphi \le 2\pi$$

Laplaceov operátor $\Delta u = u_{xx} + u_{yy}$ v polárnych súradniciach

$$u_x = u_r r_x + u_\varphi \varphi_x$$
$$u_{xx} = u_{rr}(r_x)^2 + 2u_{r\varphi} r_x \varphi_x + u_{\varphi\varphi}(\varphi_x)^2 + u_r r_{xx} + u_\varphi \varphi_{xx}$$
$$u_{yy} = u_{rr}(r_y)^2 + 2u_{r\varphi} r_y \varphi_y + u_{\varphi\varphi}(\varphi_y)^2 + u_r r_{yy} + u_\varphi \varphi_{yy}$$

Pritom $r = \sqrt{x^2+y^2}$ $\qquad \varphi = \operatorname{arctg}\frac{y}{x}$

$$r_x = \frac{x}{\sqrt{x^2+y^2}} \qquad (r_x)^2 = \frac{x^2}{x^2+y^2}$$

$$\varphi_x = \frac{1}{1+\frac{y^2}{x^2}} \cdot -\frac{y}{x^2} = \frac{-y}{x^2+y^2}$$

$$(r_x)^2 + (r_y)^2 = 1$$

$$\varphi_{xx} = \frac{+2xy}{(x^2+y^2)^2}$$

$$r_{xx} = \frac{\sqrt{x^2+y^2} - x\frac{x}{\sqrt{x^2+y^2}}}{x^2+y^2} = \frac{y^2}{(x^2+y^2)^{\frac{3}{2}}}$$

$$\varphi_y = \frac{1}{1+\frac{y^2}{x^2}} \; \frac{1}{x} = \frac{x}{x^2+y^2}$$

$$r_{xx} + r_{yy} = \frac{1}{(x^2+y^2)^{\frac{1}{2}}} = \frac{1}{r}$$

$$\varphi_{yy} = \frac{-2xy}{(x^2+y^2)^2}$$

$$\varphi_{xx} + \varphi_{yy} = 0$$

$$(\varphi_x)^2 + (\varphi_y)^2 = \frac{1}{x^2+y^2} = \frac{1}{r^2}$$

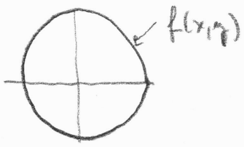
$$r_x\varphi_x + r_y\varphi_y = 0$$

$$u_{xx} + u_{yy} = u_{rr} + \frac{1}{r^2}u_{\varphi\varphi} + \frac{1}{r}u_r$$

Dirichletova úloha na kruhu

$$\Delta u = 0$$

$$u(x,y) = f(x,y) \qquad \text{pre} \qquad x^2 + y^2 = r_0^2$$



Prejde do tvaru

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\varphi\varphi} = 0$$

$$r^2 u_{rr} + r u_r + u_{\varphi\varphi} = 0$$

$$u(r_0, \varphi) = f(r_0, \varphi) = f(\varphi)$$

$$u(r, 0) = u(r, 2\pi) \qquad \text{– spojitosť}$$

Metóda separácie $\quad u(r,\varphi) = R(r)\,\phi(\varphi)$

$$r^2 R'' \phi + r R' \phi + R \phi'' = 0$$

$$\frac{r^2 R''}{R} + \frac{r R'}{R} = -\frac{\phi''}{\phi} = \lambda$$

$$\phi'' + \lambda \phi = 0 \qquad\qquad r^2 R'' + r R' - \lambda R = 0$$

$$\phi(0) = \phi(2\pi)$$

Riešme pre ϕ $\quad$ ak $\lambda = 0$ $\quad \phi(\varphi) = A_0$

ak $\lambda > 0$ $\quad \phi(\varphi) = A_1 \cos\sqrt{\lambda}\varphi + B \sin\sqrt{\lambda}\varphi \qquad \sqrt{\lambda} = k$

$$\phi_k(\varphi) = A_k \cos k\varphi + B_k \sin k\varphi$$

Riešenie pre R ak $\lambda_k = k^2$

$$r^2 R'' + r R' - k^2 R = 0$$

pre $k = 0$ dostávame rovnicu $\quad r R'' + R' = 0$

$$\frac{R''}{R'} = -\frac{1}{r}$$

$$\ln R' = -\ln r + c$$

$$R_0(r) = d_0 \qquad R' = \tilde{c}\frac{1}{r} \qquad R = \tilde{c} \ln r + d$$

pre $k>0$ $\qquad r^2R'' + rR' - k^2R = 0$

Použijeme substitúciu $\quad r = e^t$

$$\tilde{R}'(t) = R'(r)\cdot r$$
$$\tilde{R}''(t) = R''(r)\cdot r^2 + R'(r)\cdot r$$

$$\tilde{R}''(t) - k^2\tilde{R}(t) = 0 \qquad \text{Char.} \quad \lambda^2 - k^2 = 0$$

$$\tilde{R}_k(t) = c_k e^{kt} + d_k e^{-kt}$$
$$R_k(r) = c_k r^k + d_k r^{-k}$$

aby bolo riešenie definované aj v $r=0$ kladieme $d_k = 0$

$$u(r,\varphi) = a_0 + \sum_{k=1}^{\infty} a_k r^k \cos k\varphi + b_k r^k \sin k\varphi$$

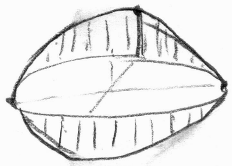
$$u(r_0,\varphi) = a_0 + \sum_{k=1}^{\infty} a_k r_0^k \cos k\varphi + b_k r_0^k \sin k\varphi = f(\varphi)$$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\varphi)\,d\varphi \qquad a_k = \frac{1}{r_0^k}\frac{1}{\pi}\int_0^{2\pi} f(\varphi)\cos k\varphi\,d\varphi$$

$$b_k = \frac{1}{r_0^k}\frac{1}{\pi}\int_0^{2\pi} f(\varphi)\sin k\varphi\,d\varphi$$

Príklad $\qquad \Delta u = 0$

$u(x,y) = f(x,y)$ pre $x^2+y^2=1$ a $f(x,y) = f(\varphi) = \sin\varphi$



Riešenie $\quad u(r,\varphi) = a_0 + \sum_{k=1}^{\infty} a_k r^k \cos k\varphi + b_k r^k \sin k\varphi = r\sin\varphi$

a $\quad u(1,\varphi) = \sin\varphi \qquad$ teda $\quad a_0 = \frac{1}{2\pi}\int_0^{2\pi}\sin\varphi\,d\varphi = 0$

$$a_k = \frac{1}{\pi}\int_0^{2\pi}\sin\varphi\cos k\varphi\,d\varphi = 0 \qquad b_k = 0 \text{ pre } k>1$$

$$b_1 = \frac{1}{\pi}\int_0^{2\pi}\sin^2\varphi\,d\varphi = \frac{1}{\pi}\,\frac{1}{2}[\varphi]_0^{2\pi} = 1$$