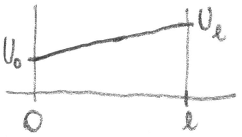
2. Nehomogénne okrajové podmienky

Riešme úlohu
$$u_t = c\,u_{xx}$$
$$u(0,t) = U_0 \qquad u(l,t) = U_l \qquad\qquad U_0, U_l \text{ - konštanty}$$
$$u(x,0) = \varphi(x)$$

Hľadajme riešenie v tvare súčtu dvoch funkcií
$$u(x,t) = v(x,t) + w(x,t)$$
z ktorých $w(x,t)$ spĺňa nehomogénne OP a $v(x,t)$ homogénne

Zvoľme $w(x,t)$ čo najjednoduchšie



$$w(x,t) = w(x) = U_0 \cdot \frac{l-x}{l} + U_l \frac{x}{l} = \frac{1}{l}\left(U_0(l-x) + U_l x\right)$$

Zrejme $w_t = 0$

a z lineárnosti (alebo výpočtom) $w_{xx} = 0$

$$v_t = c\,v_{xx}$$
$$v(0,t) = 0 \qquad v(l,t) = 0$$
$$v(x,0) = \varphi(x) - w(x,0) = \varphi(x) - \frac{1}{l}\left(U_0(l-x) + U_l \cdot x\right)$$

To je homogénna rovnica, ktorú vieme riešiť.
Celkové riešenie je potom
$$u(x,t) = v(x,t) + w(x)$$

Príklad $u_t = u_{xx}$

$$u(0,t)=0 \qquad u(2,t)=1$$
$$u(x,0) = \sin\frac{\pi x}{2}$$

$$u(x,t) = v(x,t) + w(x,t) \qquad\qquad w(x,t) = w(x) = \frac{1}{2}x$$

$$v_t = v_{tt}$$
$$v(0,t)=0 \qquad v(2,t)=0$$
$$v(x,0) = \sin\frac{\pi x}{2} - \frac{1}{2}x$$

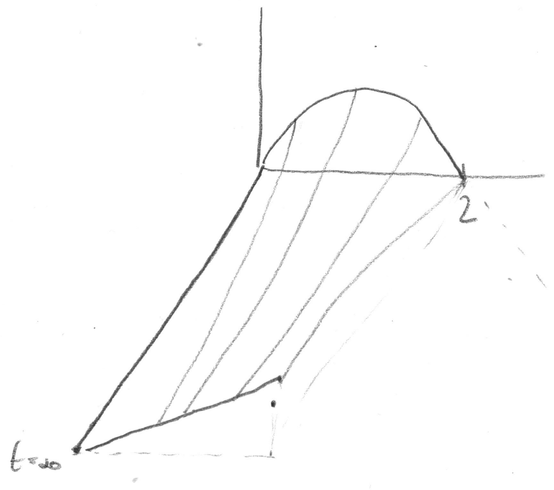
$$v(x,t) = \sum_{k=1}^{\infty} b_k \sin\frac{k\pi}{2}x\, e^{-\left(\frac{k\pi}{2}\right)^2 t}$$

$$b_k = \frac{2}{2}\int_0^2 \left(\sin\frac{\pi x}{2} - \frac{1}{2}x\right)\sin\frac{k\pi}{2}x\,dx = \int_0^2 \sin\frac{\pi}{2}x\,\sin k\frac{\pi}{2}x\,dx - \frac{1}{2}\int_0^2 x\sin\frac{k\pi}{2}x\,dx$$

Pre $k\geq 2$ $$b_k = -\frac{1}{2}\left(\left[-x\cdot\frac{\cos\frac{k\pi}{2}x}{\frac{k\pi}{2}}\right]_0^2 + \int_0^2 \frac{\cos\frac{k\pi}{2}x}{\frac{k\pi}{2}}\,dx\right) = -\frac{1}{2}\left(\frac{-2(-1)^k}{\frac{k\pi}{2}} + 0\right) = 2\frac{(-1)^k}{k\pi}$$

pre $k=1$ $$b_1 = \int_0^2 \left(\sin\frac{\pi}{2}x\right)^2 dx + 2\cdot\frac{(-1)}{\pi} = \int_0^2 \frac{1-\cos\pi x}{2}\,dx - \frac{2}{\pi} = 1 - \frac{2}{\pi}$$

$$u(x,t) = \sin\frac{\pi}{2}x\cdot e^{-\frac{\pi^2}{4}t} + \sum_{k=1}^{\infty} 2\frac{(-1)^k}{k\pi}\sin\frac{k\pi}{2}x\, e^{-\left(\frac{k\pi}{2}\right)^2 t} + \frac{1}{2}x$$



3. Nehomogénne okrajové podmienky - nekonštantné

$$u_t = c u_{xx}$$

$$u(0,t) = g_0(t) \qquad u(l,t) = g_l(t)$$

$$u(x,0) = \varphi(x)$$

Riešenie hľadáme v tvare

$$u(x,t) = v(x,t) + w(x,t)$$

Funkcia $w(x,t)$ splňa okrajové podmienky.

Položme

$$w(x,t) = g_0(t) \cdot \frac{l-x}{l} + g_l(t) \cdot \frac{x}{l}$$

Teraz $\quad w_t(x,t) = g_0'(t) \frac{l-x}{l} + g_l'(t) \frac{x}{l} \qquad w_{xx}(x,t) = 0$

Dosadením do rovnice a do podmienok

$$v_t + g_0'(t) \frac{l-x}{l} + g_l'(t) \frac{x}{l} = c v_{xx}$$

$$v(0,t) = 0 \qquad v(l,t) = 0$$

$$v(x,0) = \varphi(x) - g_0(0) \frac{l-x}{l} - g_l(0) \frac{x}{l}$$

Prepísaná rovnica $\quad v_t = c v_{xx} + f(x,t) \qquad f(x,t) = -\left(g_0'(t) \frac{l-x}{l} - g_l'(t) \frac{x}{l}\right.$

$$v(0,t) = 0 \qquad v(l,t) = 0$$

$v(x,0) = \varphi_1(x) \qquad \varphi_1(x) = \varphi(x) - g_0(0) \frac{l-x}{l} - g_l(0) \frac{x}{l}$

To je nehomogénna rovnica s homogénnymi okrajovými podmienkami a tú už vieme riešiť.

V prípade Neumanovej - nehomogénnej - podmienky

$$u_t = c\, u_{xx}$$

$u(0,t) = g_0(t)$ $\qquad u_x(l,t) = g_e(t)$

$u(x,0) = \varphi(x)$

postupujeme rovnako: $\qquad u(x,t) = v(x,t) + w(x,t)$

Len voľbu w modifikujeme

$w(0,t) = g_0(t) \qquad w_x(l,t) = g_e(t)$

dosiahneme ak $\qquad w(x,t) = g_0(t)\, \frac{l-x}{l} + G(t) \cdot \frac{x}{l}$

Zrejme $w(0,t) = g_0(t)$

$w_x(x,t) = g_0(t) \cdot \frac{-1}{l} + G(t) \cdot \frac{1}{l}$

Preto $\qquad w_x(l,t) = \quad$ — " — $\quad = g_e(t)$

a teda volíme $\qquad G(t) = g_e(t) \cdot l + g_0(t)$

$$w(x,t) = g_0(t) \cdot \frac{l-x}{l} + \left(g_e(t)\, l + g_0(t)\right)\frac{x}{l} = g_e(t) \cdot x + g_0(t)$$

Dosadením do rovnice je

$v_t + g_e'(t) \cdot x + g_0'(t) = c\, v_{xx}$ $\qquad v_t = c\, v_{xx} + f(x,t) \qquad f(x,t) = -g_e'(t)x - g_0'(t)$

$v(0,t) = 0 \qquad v_x(l,t) = 0$

$v(x,0) = \varphi(x) - g_e(0)\, x - g_0(0)$ $\qquad v(x,0) = \varphi_1(x)$

Príklad $\qquad u_t = u_{xx}$ $\qquad c = 1$

$u(0,t) = 0 \qquad u_x(\pi,t) = \sin t$ $\qquad l = \pi \qquad g_0(t) = 0$

$u(x,0) = 1$ $\qquad g_e(t) = \sin x$

$w(x,t) = \sin t \cdot x$ $\qquad v(x,t)$ spĺňa úlohu

$v_t = v_{xx} - \cos t \cdot x$

$v(0,t) = 0 \qquad v_x(\pi,t) = 0$

$v(x,0) = 1$

$$v(x,t) = X(x) \cdot T(t)$$

$$X'' + \lambda X = 0$$

$$X(0) = 0 \qquad X'(\pi) = 0$$

$$X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x \qquad A = 0 \qquad B\sqrt{\lambda}\cos\sqrt{\lambda}\pi = 0$$

$$\sqrt{\lambda}\pi = \frac{\pi}{2} + k\pi$$

$$\lambda_k = \left(\frac{1}{2} + k\right)^2$$

$$X_k(x) = B_k \sin\left(\frac{1}{2} + k\right)x$$

$$v(x,t) = \sum_{k=0}^{\infty} T_k(t) \cdot \sin\left(\frac{1}{2} + k\right)x$$

$$f(x,t) = -\cos t \cdot x = \sum_{k=0}^{\infty} f_k(t) \cdot \sin\left(\frac{1}{2} + k\right)x$$

$$f_k(t) = \frac{2}{\pi}\int_0^{\pi} -\cos t \cdot x \sin\left(\frac{1}{2} + k\right)x \, dx =$$

$$= -\frac{2\cos t}{\pi}\left(\left[x \cdot \frac{-\cos\left(\frac{1}{2}+k\right)x}{\frac{1}{2}+k}\right]_0^{\pi} + \int_0^{\pi} \frac{\cos\left(\frac{1}{2}+k\right)x}{\frac{1}{2}+k} dx\right) =$$

$$= -\frac{2\cos t}{\pi}\left[\frac{\sin\left(\frac{1}{2}+k\right)x}{\left(\frac{1}{2}+k\right)^2}\right]_0^{\pi} = -\frac{2\cos t}{\pi}\frac{(-1)^k}{\left(\frac{1}{2}+k\right)^2}$$

$$\varphi_1(x) = 1 = \sum_{k=0}^{\infty} \varphi_k \sin\left(\frac{1}{2}+k\right)x$$

$$\varphi_k = \frac{2}{\pi}\int_0^{\pi} 1 \cdot \sin\left(\frac{1}{2}+k\right)x \, dx = \frac{2}{\pi}\left[-\frac{\cos\left(\frac{1}{2}+k\right)x}{\frac{1}{2}+k}\right]_0^{\pi} =$$

$$= \frac{2}{\pi}\frac{1}{\frac{1}{2}+k}$$

Riešime systém rovnic

$$T_k' + \left(\frac{1}{2}+k\right)^2 T_k = -\frac{2\cos t}{\pi}\frac{(-1)^k}{\left(\frac{1}{2}+k\right)^2}$$

$$T_k(0) = \frac{2}{\pi}\frac{1}{\frac{1}{2}+k}$$

Homogénna časť

$$T_{k(hom)}(t) = \frac{2}{\pi}\cdot\frac{1}{\frac{1}{2}+k}\cdot e^{-\left(\frac{1}{2}+k\right)^2 t}$$

Partikulárne riešenie

$$T_{k(part)}(t) = \int_0^t -\frac{2\cos s}{\pi}\,\frac{(-1)^k}{\left(\frac{1}{2}+k\right)^2}\cdot e^{\left(\frac{1}{2}+k\right)^2 s}\,ds \cdot e^{-\left(\frac{1}{2}+k\right)^2 t}$$

$$= \frac{(-1)^{k+1}\cdot 2}{\pi\left(\frac{1}{2}+k\right)^2}\int_0^t \cos s\, e^{\left(\frac{1}{2}+k\right)^2 s}\,ds \cdot e^{-\left(\frac{1}{2}+k\right)^2 t}$$

Zvlášť integrujme

$$\int \cos s\, e^{ms}\,ds = \sin s\cdot e^{ms} - \int \sin s\cdot m e^{ms}\,ds =$$

$$= \sin s\, e^{ms} - \left(-\cos s\, m e^{ms} + \int \cos s\, m^2 e^{ms}\,ds\right) =$$

$$= (\sin s + m\cos s)\,e^{ms} - m^2\int \cos s\, e^{ms}\,ds$$

$$(1+m^2)\int \cos s\, e^{ms}\,ds = (\sin s + m\cos s)\,e^{ms}$$

$$\int \cos s\, e^{ms}\,ds = \frac{1}{1+m^2}(\sin s + m\cos s)\,e^{ms} =$$

$$\int_0^t \cos s\, e^{\left(\frac{1}{2}+k\right)^2 s}\,ds = \left[\frac{1}{1+\left(\frac{1}{2}+k\right)^4}\left(\sin s + \left(\tfrac{1}{2}+k\right)^2\cos s\right) e^{\left(\frac{1}{2}+k\right)^2 s}\right]_0^t =$$

$$= \frac{1}{1+\left(\frac{1}{2}+k\right)^4}\left(\left(\sin t + \left(\tfrac{1}{2}+k\right)^2\cos t\right) e^{\left(\frac{1}{2}+k\right)^2 t} - \left(\tfrac{1}{2}+k\right)^2\right)$$

$$T_k(t) = \frac{2}{\pi}\,\frac{1}{\frac{1}{2}+k}\,e^{-\left(\frac{1}{2}+k\right)^2 t} + \frac{(-1)^{k+1}\cdot 2}{\pi\left(\frac{1}{2}+k\right)^2}\,\frac{1}{1+\left(\frac{1}{2}+k\right)^4}\left(\sin t + \left(\tfrac{1}{2}+k\right)^2\cos t - \left(\tfrac{1}{2}+k\right)^2 e^{-\left(\frac{1}{2}+k\right)^2 t}\right)$$

$$u(x,t) = v(x,t) + w(x,t) =$$

$$= \sum_{k=0}^{\infty}\left[\frac{2}{\pi}\left(\frac{1}{\frac{1}{2}+k} + \frac{(-1)^k}{1+\left(\frac{1}{2}+k\right)^4}\right)e^{-\left(\frac{1}{2}+k\right)^2 t} + \frac{2(-1)^{k+1}}{\pi\left(\frac{1}{2}+k\right)^2\left(1+\left(\frac{1}{2}+k\right)^4\right)}\left(\sin t + \left(\tfrac{1}{2}+k\right)^2\cos t\right)\right]\sin\left(\tfrac{1}{2}+k\right)x +$$

$$+ \sin t\,.\,x$$