Difúzna rovnica na konečnom intervale

Rovnica vedenia tepla

$$u_t = c u_{xx} \qquad 0 \leq x \leq l \qquad t \geq 0$$

$u(x,t)$ – teplota v mieste x a v čase t

$c > 0$

Dirichletove okrajové podmienky

$$u(0,t) = 0 \qquad u(l,t) = 0$$

Začiatočné rozloženie

$$u(x,0) = \varphi(x)$$

Riešenie metódou separácie

$$u(x,t) = X(x) \cdot T(t)$$

$$T'X = c X'' T$$

$$\frac{X''}{X} = \frac{1}{c}\frac{T'}{T} = -\lambda$$

$$X'' + \lambda X = 0 \qquad \text{a} \qquad T' + \lambda c T = 0$$

$$X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$$

$$X(0) = 0 \qquad X(l) = 0$$

$$A = 0 \qquad \sin\sqrt{\lambda}l = 0 \qquad \lambda = \left(\frac{k\pi}{l}\right)^2$$

$$X_k(x) = B_k \sin\frac{k\pi}{l}x$$

$$T(t) = Ce^{-\lambda c t} \qquad T_k(t) = C_k e^{-\left(\frac{k\pi}{l}\right)^2 ct}$$

$$u(x,t) = \sum_{k=1}^{\infty} a_k e^{-\left(\frac{k\pi}{l}\right)^2 ct} \sin\frac{k\pi}{l}x$$

$$u(x,0)=\sum_{k=1}^{\infty} a_k \sin\frac{k\pi}{\ell}x = \varphi(x)$$

$$a_k=\frac{2}{\ell}\int_0^{\ell}\varphi(x)\sin\frac{k\pi}{\ell}x\,dx$$

Príklad 1. $u_t = 2u_{xx}$

$u(0,t)=0 \qquad u(1,t)=0 \qquad \ell=1$

$u(x,0)=x^2(1-x)$

$$u(x,t)=\sum_{k=1}^{\infty} a_k e^{-k^2\pi^2 2t}\sin k\pi x$$

$$a_k = 2\int_0^1 x^2(1-x)\sin k\pi x\,dx =$$

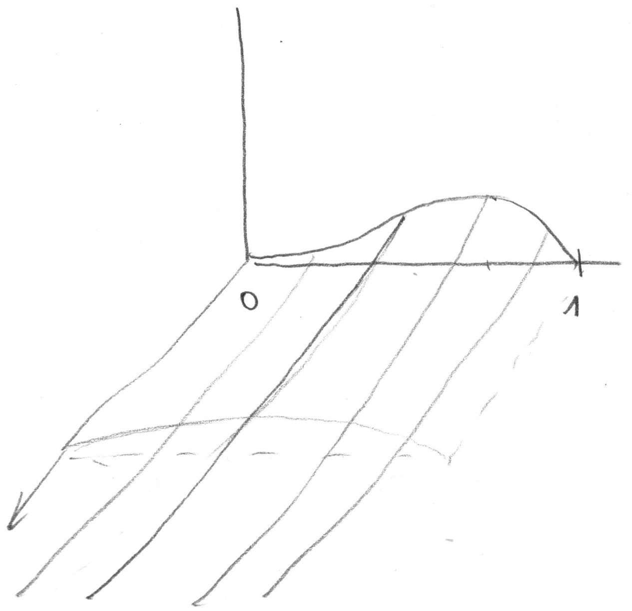
$$=2\left(\left[(x^2-x^3)\cdot\left(-\frac{\cos k\pi x}{k\pi}\right)\right]_0^1+\int_0^1\frac{\cos k\pi x}{k\pi}\cdot(2x-3x^2)\,dx\right)=$$

$$=2\left(\left[(2x-3x^2)\cdot\frac{\sin k\pi x}{k^2\pi^2}\right]_0^1-\int_0^1\frac{\sin k\pi x}{k^2\pi^2}(2-6x)\,dx\right)=$$

$$=2\left(\left[(2-6x)\cdot\left(-\frac{\cos k\pi x}{k^3\pi^3}\right)\right]_0^1+\int_0^1\frac{\cos k\pi x}{k^3\pi^3}(-6)\,dx\right)=$$

$$=2\left(\frac{4\cos k\pi-2}{k^3\pi^3}+0\right)=4\frac{2(-1)^k-1}{k^3\pi^3}$$

$$u(x,t)=\sum_{k=1}^{\infty}4\,\frac{2(-1)^k-1}{k^3\pi^3}\,e^{-2k^2\pi^2t}\sin k\pi x.$$



Difúzna rovnica s Neumanovými podmienkami

$$u_t = c u_{xx} \qquad c>0$$

$$u_x(0,t)=0 \qquad u_x(l,t)=0$$

$$u(x,0)=\varphi(x)$$

Pre riešenie $u(x,t)=X(x)\cdot T(t)$ dostaneme separáciou

$$X''+\lambda X=0 \qquad T'+\lambda c T=0$$

$$X'(0)=0 \qquad X'(l)=0$$

Rovnako, ako vo vlnovej rovnici pribudne pre $\lambda=0$ riešenie

$$X_0(x)=A_0 \qquad \text{a k nemu} \qquad T_0(t)=C_0$$

Pre $\lambda>0$ máme

$$X(x)=A\cos\sqrt{\lambda}x+B\sin\sqrt{\lambda}x$$

a $$X'(x)=-A\sqrt{\lambda}\sin\sqrt{\lambda}x+B\sqrt{\lambda}\cos\sqrt{\lambda}x$$

$$X'(0)=B=0 \qquad X'(l)=-A\sqrt{\lambda}\sin\sqrt{\lambda}l=0 \Rightarrow \sqrt{\lambda}l=k\pi \qquad \sqrt{\lambda}=\frac{k\pi}{l}$$

$$X_k(x)=A_k\cos\frac{k\pi}{l}x \qquad \text{a} \qquad T_k(t)=C_k e^{-\left(\frac{k\pi}{l}\right)^2 ct}$$

$$u(x,t)=a_0+\sum_{k=1}^{\infty} a_k\cos\frac{k\pi}{l}x\, e^{-\left(\frac{k\pi}{l}\right)^2 ct}$$

Pritom $$a_0=\frac{1}{l}\int_0^l \varphi(x)\,dx$$

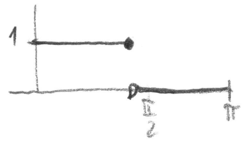
$$a_k=\frac{2}{l}\int_0^l \varphi(x)\cdot\cos\frac{k\pi}{l}x\,dx$$

Príklad $u_t = 3u_{xx}$

$u_x(0,t)=0 \qquad u_x(\pi,t)=0$

$u(x,0) = \varphi(x)$

$$\varphi(x) = \begin{cases} 1 & \text{pre} \quad x \in \langle 0, \frac{\pi}{2} \rangle \\ 0 & \text{pre} \quad x \in (\frac{\pi}{2}, \pi \rangle \end{cases}$$

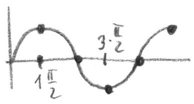


$$u(x,t) = a_0 + \sum_{k=1}^{\infty} a_k \cos kx \cdot e^{-3k^2 t}$$

Pritom
$$a_0 = \frac{1}{\pi}\int_0^{\pi} \varphi(x)\,dx = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} 1\,dx = \frac{1}{2}$$

$$a_k = \frac{2}{\pi}\int_0^{\pi} \varphi(x)\cos kx\,dx = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cos kx\,dx = \frac{2}{\pi}\left[\frac{\sin kx}{k}\right]_0^{\frac{\pi}{2}} =$$

$$= \frac{2}{k\pi}\sin\frac{k\pi}{2}$$



$$= \frac{2}{k\pi}(-1)^m \qquad \text{pre} \quad k = 2m+1$$

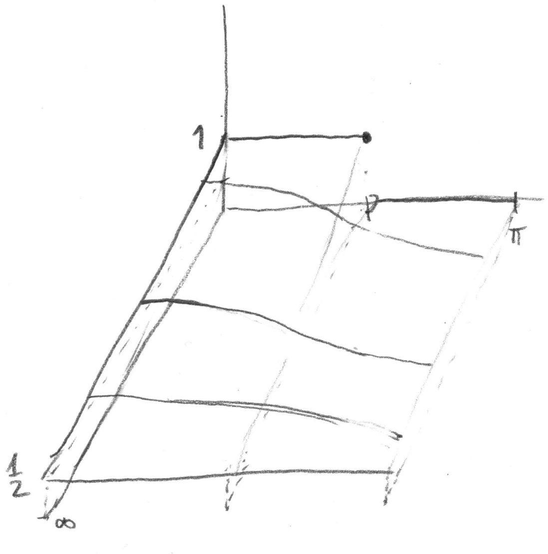
$$u(x,t) = \frac{1}{2} + \sum_{m=0}^{\infty} \frac{2}{(2m+1)\pi}(-1)^m \cos(2m+1)x\, e^{-3(2m+1)^2 t}$$

Približné riešenie pre m „idúce po 0"

$$u_0(x,t) = \frac{1}{2} + \frac{2}{\pi}\cos x\, e^{-3t}$$

pre m „idúce po 1"

$$u_1(x,t) = \frac{1}{2} + \frac{2}{\pi}\cos x\, e^{-3t} - \frac{2}{3\pi}\cos 3x\, e^{-27t}$$



Difúzna rovnica so zmiešanou podmienkou

$u_t = c\,u_{xx}$ $\qquad c>0$

$u(0,t)=0 \qquad u(l,t)=h\,u_x(l,t) \qquad h\neq 0$

$u(x,0)=\varphi(x)$

Riešenie $\quad u(x,t)=X(x)\,T(t)$

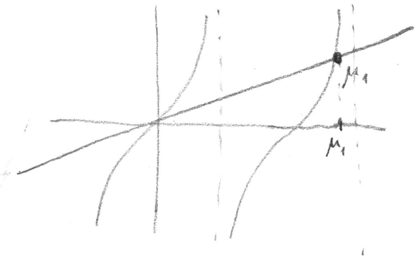
$X'' + \lambda X = 0 \qquad T' + \lambda c T = 0$

$X(0)=0 \qquad X(l)=h\,X'(l)$

$X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$

$X(0)=A=0 \qquad X(l)=hX'(l) \Rightarrow \sin\sqrt{\lambda}l = h\sqrt{\lambda}\cos\sqrt{\lambda}l$

$\operatorname{tg}\sqrt{\lambda}l = h\sqrt{\lambda}$



μ_k - korene (kladné)

$\lambda_k = \left(\frac{\mu_k}{l}\right)^2$

$X_k(x) = B_k \sin\frac{\mu_k}{l}x \qquad T_k(t) = C_k e^{-\left(\frac{\mu_k}{l}\right)^2 ct}$

$$u(x,t) = \sum_{k=1}^{\infty} b_k \sin\frac{\mu_k}{l}x\, e^{-\left(\frac{\mu_k}{l}\right)^2 ct}$$

Pritom $\quad b_k = \frac{2}{\int_0^l \left(\sin\frac{\mu_k}{l}x\right)^2 dx} \int_0^l \varphi(x)\sin\frac{\mu_k}{l}x\,dx =$

$$b_k = \frac{2}{l}\,\frac{1}{1-\frac{\sin 2\mu_k}{2\mu_k}} \int_0^l \varphi(x)\sin\frac{\mu_k}{l}x\,dx$$

Príklad $u_t = 2 u_{xx}$ $c = 2$

$u(0,t) = 0$ $u(\pi,t) = u_x(\pi,t)$ $h = 1$ $l = \pi$

$u(x,0) = x$

Riešenie $u(x,t) = \sum_{k=1}^{\infty} b_k \sin \frac{\mu_k}{\pi} x \, e^{-\left(\frac{\mu_k}{\pi}\right)^2 2t}$

Pritom μ_k sú korene rovnice $\operatorname{tg} y = \frac{h}{l} y = \frac{1}{\pi} y$

Vypočítajme niekoľko koreňov μ_k $\mu_1 \doteq 4.053$

$\mu_2 \doteq 7.455$

$\mu_3 \doteq 10.710$

Príslušné koeficienty b_k sú $b_1 = \frac{2}{\pi} \frac{1}{1 - \frac{\sin 2\mu_1}{2\mu_1}} \int_0^{\pi} x \sin \frac{\mu_1}{\pi} x \, dx =$

$$= \frac{2}{\pi} \frac{1}{1 - \frac{\sin 2\mu_1}{2\mu_1}} \left(\left[\frac{-\cos \frac{\mu_1}{\pi} x}{\frac{\mu_1}{\pi}} \cdot x \right]_0^{\pi} + \int_0^{\pi} \frac{\cos \frac{\mu_1}{\pi} x}{\frac{\mu_1}{\pi}} dx \right) =$$

$$= \frac{2}{\pi} \frac{1}{1 - \frac{\sin 2\mu_1}{2\mu_1}} \left(-\frac{\pi^2}{\mu_1} \cos \mu_1 + \left(\frac{\pi}{\mu_1}\right)^2 \left[\sin \frac{\mu_1}{\pi} x \right]_0^{\pi} \right) = \frac{2}{\pi} \frac{1}{1 - \frac{\sin 2\mu_1}{2\mu_1}} \frac{\pi^2}{\mu_1} \left(\frac{\sin \mu_1}{\mu_1} - \cos \mu_1 \right) =$$

$$= 2 \frac{1}{1 - \frac{\sin 2\mu_1}{2\mu_1}} \frac{\pi}{\mu_1} \left(\frac{1}{\pi} - 1 \right) \cos \mu_1 \doteq 0.7352$$

$b_2 \doteq -0.2344$

$b_3 \doteq 0.1156$

Približné riešenie

$$u(x,t) \doteq 0.7352 \cdot \sin \frac{4.053}{\pi} x \, e^{-3.329 t} - 0.2344 \sin \frac{7.455}{\pi} x \cdot e^{-11.26 t} +$$

$$+ 0.1156 \cdot \sin \frac{10.710}{\pi} x \cdot e^{-23.24 t}$$

Problém: Nájdite nasledujúci sčítanec

Nehomogénne úlohy

1 Nehomogénna rovnica

Uvažujme difúznu rovnicu so vstupom

$$u_t = c\,u_{xx} + f(x,t)$$

$$u(0,t)=0 \qquad u(\ell,t)=0$$

$$u(x,0)=\varphi(x)$$

Riešenie v dvoch krokoch. 1. krok Homogénna úloha

$$u_t = c\,u_{xx}$$

$$u(0,t)=0 \qquad u(\ell,t)=0$$

$$u(x,t)=X(x).T(t)$$

$$X''+\lambda X=0$$

$$X(0)=0 \qquad X(\ell)=0 \qquad \lambda_k=\left(\frac{k\pi}{\ell}\right)^2$$

Riešenie $X_k = \sin\frac{k\pi}{\ell}x$

Teda

$$u(x,t)=\sum_{k=1}^{\infty} T_k(t)\sin\frac{k\pi}{\ell}x$$

2 krok Rozvinieme vstup $f(x,t)$ do sínusového radu v premennej x

$$f(x,t)=\sum_{k=1}^{\infty} f_k(t).\sin\frac{k\pi}{\ell}x$$

Pritom $$f_k(t)=\frac{2}{\ell}\int_0^{\ell} f(x,t).\sin\frac{k\pi}{\ell}x\,dx$$

A tiež $$\varphi(x)=\sum_{k=1}^{\infty} a_k \sin\frac{k\pi}{\ell}x$$

s $$a_k=\frac{2}{\ell}\int_0^{\ell}\varphi(x)\sin\frac{k\pi}{\ell}x\,dx$$

Dosadíme do pôvodnej rovnice $\quad u_t = c u_{xx} + f(x,t)$

$$\sum_{k=1}^{\infty} T_k'(t) \sin\frac{k\pi}{l}x = -c\sum_{k=1}^{\infty} T_k(t)\left(\frac{k\pi}{l}\right)^2 \sin\frac{k\pi}{l}x + \sum_{k=1}^{\infty} f_k(t)\sin\frac{k\pi}{l}x$$

a podmienky $\quad u(x,0) = \varphi(x)$

$$\sum_{k=1}^{\infty} T_k(0)\cdot \sin\frac{k\pi}{l}x = \sum_{k=1}^{\infty} a_k \sin\frac{k\pi}{l}x$$

Pre každé $k \in N$ dostávame ODR 1. rádu

$$T_k' = -c\left(\frac{k\pi}{l}\right)^2 T_k + f_k(t)$$

s podmienkou $\quad T_k(0) = a_k$

Jej riešenie je $\quad T_k(t) = a_k e^{-c\left(\frac{k\pi}{l}\right)^2 t} + \int_0^t f_k(s)\cdot e^{c\left(\frac{k\pi}{l}\right)^2 s} ds \cdot e^{-c\left(\frac{k\pi}{l}\right)^2 t}$

Riešenie pôvodnej nehomogénnej úlohy je

$$\underline{u(x,t) = \sum_{k=1}^{\infty}\left(a_k + \int_0^t f_k(s) e^{c\left(\frac{k\pi}{l}\right)^2 s} ds\right) e^{-c\left(\frac{k\pi}{l}\right)^2 t}\cdot \sin\frac{k\pi}{l}x}$$

Príklad $\quad u_t = 2u_{xx} + xt \qquad c=2$

$u(0,t) = 0 \qquad u(1,t) = 0 \qquad l = 1$

$u(x,0) = 1$

1. krok $\quad X_k = \sin k\pi x$

2 krok

$$f_k(t) = 2\int_0^1 xt \sin k\pi x\,dx = 2t\left(\left[-x\frac{\cos k\pi x}{k\pi}\right]_0^1 + \int_0^1 \frac{\cos k\pi x}{k\pi}\right) = -2t\frac{(-1)^k}{k\pi}$$

$$a_k = 2\int_0^1 1 \sin k\pi x\,dx = 2\cdot\left[-\frac{\cos k\pi x}{k\pi}\right]_0^1 = 2\,\frac{1-(-1)^k}{k\pi}$$

Riešime rovnice

$$T_k' = -2k^2\pi^2 T_k + 2t\frac{(-1)^{k+1}}{k\pi}$$
$$T_k(0) = 2\,\frac{1-(-1)^k}{k\pi}$$

Riešenia $$T_k(t) = \frac{2}{k\pi}(1-(-1)^k)\,e^{-2k^2\pi^2 t} + \int_0^t 2s\frac{(-1)^{k+1}}{k\pi}\,e^{2k^2\pi^2 s}\,ds \cdot e^{-2k^2\pi^2 t} =$$

$$= \frac{2}{k\pi}\left(1-(-1)^k\right)e^{-2k^2\pi^2 t} + \frac{2(-1)^{k+1}}{k\pi}\left(\left[s\cdot\frac{1}{2k^2\pi^2}e^{2k^2\pi^2 s}\right]_0^t - \int_0^t \frac{1}{2k^2\pi^2}e^{2k^2\pi^2 s}\,ds\right)e^{-2k^2\pi^2 t} =$$

$$= \frac{2}{k\pi}\left(1-(-1)^k\right)e^{-2k^2\pi^2 t} + \frac{2}{k\pi}(-1)^{k+1}\left(t\cdot\frac{1}{2k^2\pi^2} - \frac{1}{4k^4\pi^4}\left(1-e^{-2k^2\pi^2 t}\right)\right)$$

$$u(x,t) = \sum_{k=1}^{\infty}\left\{(-1)^{k+1}\left(\frac{t}{k^3\pi^3} - \frac{1}{2k^5\pi^5}\right) + \underbrace{\left(\frac{2}{k\pi}(1-(-1)^k) + \frac{1}{2k^5\pi^5}(-1)^{k+1}\right)e^{-2k^2\pi^2 t}}_{\downarrow\ 0 \text{ pre } t\to\infty}\right\}\sin k\pi x$$

$$u(x,t) \doteq \left(\frac{t}{\pi^3} - \frac{1}{2\pi^5}\right)\sin\pi x + \left(\frac{4}{\pi} + \frac{1}{\pi^5}\right)e^{-2\pi^2 t}\sin\pi x - \left(\frac{t}{8\pi^3} - \frac{1}{64\pi^5}\right)\sin 2\pi x - \frac{1}{64\pi^5}e^{-8\pi^2 t}\sin k\pi x$$