Iné typy okrajových podmienok

Okrem vlnovej rovnice s upevnenými koncami môžeme uvažovať aj rovnicu s jedným, alebo oboma voľnými koncami. Príslušná podmienka má podobu $u_x(0,t)=0$ alebo $u_x(l,t)=0$. Tento typ podmienky sa volá Neumannova podmienka.

Úloha

$$u_{tt} = c^2 u_{xx}$$

$$u(0,t)=0 \qquad u_x(l,t)=0$$

$$u(x,0)=\varphi(x) \qquad u_t(x,0)=\psi(x)$$

Riešenie prebieha rovnako, hľadáme $u(x,t)$ v podobe

$$u(x,t) = X(x).T(t)$$

a dosadením do rovnice a následnou separáciou dostaneme

$$T''X = c^2 T X''$$

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = -\lambda$$

ODR $\quad X''+\lambda X=0 \quad$ má všeobecné riešenie pre $\lambda>0$

$$X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$$

Z podmienok

$$X(0)T(t)=0 \qquad X'(l)T(t)=0$$

máme

$$X(0)=0 \qquad X'(l)=0$$

a teda

$$X(0) = A = 0$$

$$X'(x) = B\sqrt{\lambda}\cos\sqrt{\lambda}x$$

$$X'(l) = B\sqrt{\lambda}\cos\sqrt{\lambda}l = 0$$

preto $\sqrt{\lambda}l = \frac{\pi}{2}+k\pi \qquad k\in \mathbb{N}_0$

$$\lambda_k = \left(\frac{\frac{\pi}{2}+k\pi}{l}\right)^2$$

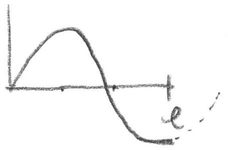
Riešenie má teraz podobu

$$X_k(x) = B_k \sin \frac{\pi + 2k\pi}{2l} x$$

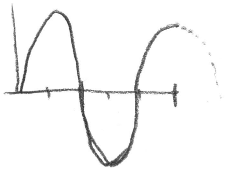
Obrázková predstava:

$k=0$ $\quad X_0(x)$

$k=1$ $\quad X_1(x)$



$k=2$ $\quad X_2(x)$



Ku každému $\lambda_k = \left(\frac{\pi+2k\pi}{2l}\right)^2$ a $X_k(x)$ nájdeme $T_k(t)$ z rovnice

$$T_k'' + \lambda_k c^2 T_k = 0$$

$$T_k(t) = C_k \sin c \frac{\pi+2k\pi}{2l} t + D_k \cos c \frac{\pi+2k\pi}{2l} t$$

Všeobecné riešenie je

$$u(x,t) = \sum_{k=0}^{\infty} u_k(x,t) = \sum_{k=0}^{\infty} \left[a_k \cos\left(\left(\frac{1+2k}{2}\right) \frac{c\pi}{l} t \right) + b_k \sin \frac{1+2k}{2} \frac{c\pi}{l} t \right] \sin \frac{1+2k}{2} \frac{\pi}{l} x$$

Splnené začiatočné podmienky zabezpečíme z rovností

$$u(x,0) = \sum_{k=0}^{\infty} a_k \sin \frac{1+2k}{2} \frac{\pi}{l} x = \varphi(x)$$

$$\int_0^l a_k \left(\sin \frac{1+2k}{2} \frac{\pi}{l} x \right)^2 dx = \int_0^l \varphi(x) \sin \frac{1+2k}{2} \frac{\pi}{l} x \, dx$$

$$a_k = \frac{2}{l} \int_0^l \varphi(x) \sin\left(\frac{1}{2}+k\right) \frac{\pi}{l} x \, dx$$

$$u_t(x,0) = \sum_{k=0}^{\infty} b_k \frac{1+2k}{2} \frac{c\pi}{l} \cdot \sin \frac{1+2k}{2} \frac{\pi}{l} x$$

$$b_k = \frac{4}{1+2k} \frac{1}{c\pi} \int_0^l \psi(x) \sin\left(\frac{1}{2}+k\right) \frac{\pi}{l} x$$

Príklad $\quad u_{tt} = 4u_{xx}$

$$u(0,t)=0 \qquad u_x(\pi,t)=0$$

$$u(x,0)=(\pi-x)\sin x \qquad u_t(x,0)=0 \qquad c=2 \quad l=\pi$$

Riešenie $\quad u(x,t)=\sum_{k=0}^{\infty}\left(a_k\cos\left(\tfrac{1}{2}+k\right)2t+b_k\sin\left(\tfrac{1}{2}+k\right)2t\right)\sin\left(\tfrac{1}{2}+k\right)x$

$b_k=0$

$$a_k=\frac{2}{\pi}\int_0^{\pi}(\pi-x)\sin x\cdot\sin\left(\tfrac{1}{2}+k\right)x\,dx=$$

$$=\frac{1}{\pi}\int_0^{\pi}\underbrace{(\pi-x)}_{g}\underbrace{\left[\cos\left(\tfrac{1}{2}-k\right)x-\cos\left(\tfrac{3}{2}+k\right)x\right]}_{f'}dx=$$

$$=\frac{1}{\pi}\left(\left[\left(\frac{\sin\left(\frac{1}{2}-k\right)x}{\frac{1}{2}-k}-\frac{\sin\left(\frac{3}{2}+k\right)x}{\frac{3}{2}+k}\right)(\pi-x)\right]_0^{\pi}+\int_0^{\pi}\frac{\sin\left(\frac{1}{2}-k\right)x}{\frac{1}{2}-k}-\frac{\sin\left(\frac{3}{2}+k\right)x}{\frac{3}{2}+k}\,dx\right)$$

$$=\frac{1}{\pi}\left[-\frac{\cos\left(\frac{1}{2}-k\right)x}{\left(\frac{1}{2}-k\right)^2}+\frac{\cos\left(\frac{3}{2}+k\right)x}{\left(\frac{3}{2}+k\right)^2}\right]_0^{\pi}=\frac{1}{\pi}\left(\frac{1}{\left(\frac{3}{2}+k\right)^2}-\frac{1}{\left(k-\frac{1}{2}\right)^2}\right)$$

$$u(x,t)=\sum_{k=0}^{\infty}\frac{1}{\pi}\left(\frac{1}{\left(\frac{3}{2}+k\right)^2}-\frac{1}{\left(k-\frac{1}{2}\right)^2}\right)\cos(2k+1)t\,\sin\left(k+\tfrac{1}{2}\right)x$$

Úloha s voľnými koncami

$$u_{tt} = c^2 u_{xx}$$

$$u_x(0,t) = 0 \qquad u_x(l,t) = 0$$

$$u(x,0) = \varphi(x) \qquad u_t(x,0) = \psi(x)$$

Riešenie je opäť v tvare $u(x,t) = X(x)\, T(t)$
a separáciou dostaneme

$$X'' + \lambda X = 0 \qquad T'' + \lambda c^2 T = 0$$

Podmienky

$$X'(0) = 0 \qquad X'(l) = 0$$

Teraz pribudne nenulové riešenie aj pre $\lambda = 0$

$$X(x) = A + Bx$$

$$X'(x) = B \qquad X'(0) = B = 0 = X'(l)$$

Teda $X_0(x) = A_0$ je konštantné riešenie

a k nemu $T'' = 0$ dáva $T_0(t) = C_0 + D_0 t$

Teda

$$u_0(x,t) = A_0 C_0 + A_0 D_0 t$$

Pre $\lambda > 0$ máme

$$X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$$

$$X'(x) = -A\sqrt{\lambda}\sin\sqrt{\lambda}x + B\sqrt{\lambda}\cos\sqrt{\lambda}x$$

$$X'(0) = B\sqrt{\lambda} = 0$$

$$X'(l) = -A\sqrt{\lambda}\sin\sqrt{\lambda}l = 0 \quad \Rightarrow \quad \sqrt{\lambda}l = k\pi \quad \lambda = \left(\frac{k\pi}{l}\right)^2$$

Riešenia $X_k(x) = A_k \cos\frac{k\pi}{l}x$

Teraz $T_k(t) = C_k \cos\frac{k\pi c}{l}t + D_k \sin\frac{k\pi c}{l}t$

a $$u(x,t) = u_0(x,t) + \sum_{k=1}^{\infty} u_k(x,t) =$$

$$= a_0 + b_0 t + \sum_{k=1}^{\infty}\left(a_k \cos\frac{k\pi c}{l}t + b_k \sin\frac{k\pi c}{l}t\right)\cos\frac{k\pi}{l}x$$

Spolu s podmienkami

$$u(x,0) = a_0 + \sum_{k=1}^{\infty} a_k \cos\frac{k\pi}{l}x = \varphi(x)$$

Koeficienty a_0, a_k sú z rozvoja $\varphi(x)$ do kosínusového radu

$$a_k = \frac{2}{l}\int_0^l \varphi(x)\cos\frac{k\pi}{l}x\,dx$$

$$a_0 = \frac{1}{l}\int_0^l \varphi(x)\,dx.$$

$$u_t(x,0) = b_0 + \sum_{k=1}^{\infty} b_k\frac{k\pi c}{l}\cos\frac{k\pi}{l}x$$

$$b_k = \frac{2}{k\pi c}\int_0^l \psi(x)\cos\frac{k\pi}{l}x\,dx$$

$$b_0 = \frac{1}{l}\int_0^l \psi(x)\,dx$$

Príklad

$$u_{tt} = u_{xx}$$

$$u_x(0,t)=0 \qquad u_x(1,t)=0$$

$$u(x,0)=1 \qquad u_t(x,0)=x \qquad\qquad c=1 \qquad l=1$$

Obidve okrajové podmienky sú Neumannove.

$$u(x,t) = a_0 + b_0 t + \sum_{k=1}^{\infty} \left(a_k \cos k\pi t + b_k \sin k\pi t\right) \cos k\pi x$$

$$a_0 = \int_0^1 1\,dx = 1$$

$$a_k = 2\int_0^1 1 \cos k\pi x\,dx = \frac{2}{k\pi}\Big[\sin k\pi x\Big]_0^1 = 0$$

$$b_0 = \int_0^1 x\,dx = \frac{1}{2}$$

$$b_k = \frac{2}{k\pi}\int_0^1 x \cos k\pi x\,dx = \frac{2}{k\pi}\left(\left[x\,\frac{\sin k\pi x}{k\pi}\right]_0^1 - \int_0^1 \frac{\sin k\pi x}{k\pi}\,dx\right) =$$

$$= \frac{2}{k\pi}\left[\frac{\cos k\pi x}{k^2\pi^2}\right]_0^1 = \frac{2\left((-1)^k - 1\right)}{k^3\pi^3}$$

Riešenie

$$u(x,t) = 1 + \frac{1}{2}t - \sum_{k=1}^{\infty} \frac{2}{k^3\pi^3}\left(1-(-1)^k\right) \sin k\pi t \cos k\pi x$$

Zmiešaná okrajová podmienka.

Uvažujme úlohu

$$u_{tt} = c^2 u_{xx}$$

$u(0,t) = 0 \qquad u(\ell,t) = h\, u_x(\ell,t) \qquad h$ - konštanta $h \neq 0$

$u(x,0) = \varphi(x) \qquad u_t(x,0) = \psi(x)$

Metódou separácie pre funkciu $u(x,t) = X(x)\cdot T(t)$ dostaneme

$$T''X = c^2 T X''$$

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = -\lambda$$

$$X'' + \lambda X = 0 \qquad T'' + \lambda c^2 T = 0$$

a z okrajových podmienok

$$X(0) = 0 \qquad X(\ell) = h X'(\ell)$$

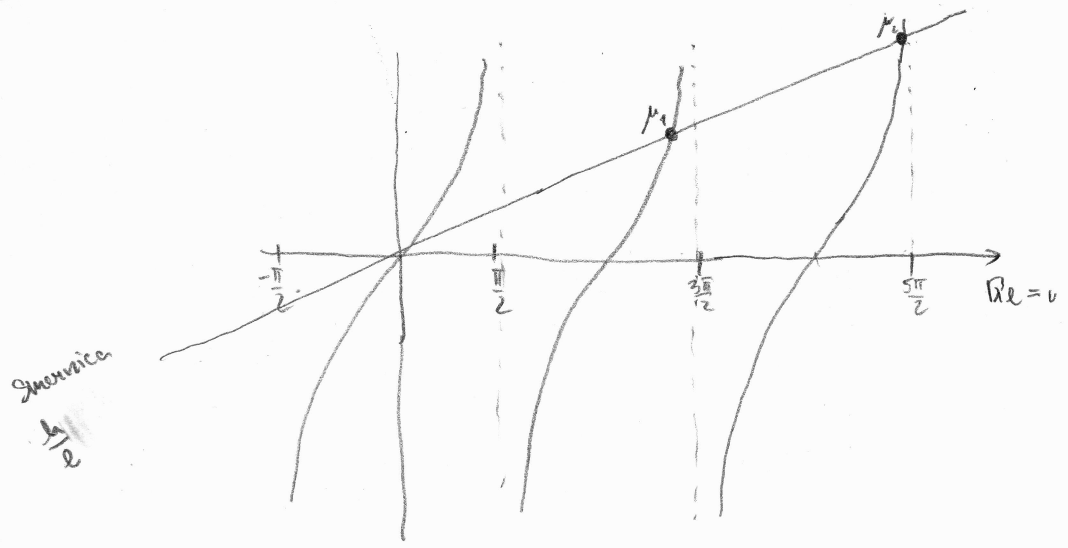
Riešenie ODR je

$$X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$$

$$X(0) = A = 0$$

$$X(\ell) = hX'(\ell) \quad \Rightarrow \quad \sin\sqrt{\lambda}\ell = h\cos\sqrt{\lambda}\ell\cdot\sqrt{\lambda}$$

$$\operatorname{tg}\sqrt{\lambda}\ell = h\sqrt{\lambda}$$



$$\sqrt{\lambda_k} = \frac{\mu_k}{\ell}$$

$$\lambda_k = \frac{\mu_k^2}{\ell^2} \qquad \lambda_1 < \lambda_2 < \lambda_3 < \dots \to \infty$$

$$X_k(x) = B_k \sin \frac{\mu_k}{\ell} x$$

Druhá rovnica $T'' + \lambda_k c^2 T = 0$ má riešenia

$$T_k(t) = C_k \cos \frac{\mu_k}{\ell} ct + D_k \sin \frac{\mu_k}{\ell} ct$$

Všeobecné riešenie je

$$u(x,t) = \sum_{k=1}^{\infty} \left(a_k \cos \frac{\mu_k}{\ell} ct + b_k \sin \frac{\mu_k}{\ell} ct \right) \sin \frac{\mu_k}{\ell} x$$

$$u(x,0) = \sum_{k=1}^{\infty} a_k \sin \frac{\mu_k}{\ell} x = \varphi(x)$$

$$a_k \int_0^\ell \sin \frac{\mu_k}{\ell} x \cdot \sin \frac{\mu_k}{\ell} x \, dx = \int_0^\ell \varphi(x) \sin \frac{\mu_k}{\ell} x \, dx$$

$$a_k = \frac{\int_0^\ell \varphi(x) \sin \frac{\mu_k}{\ell} dx}{\int_0^\ell \sin^2 \frac{\mu_k}{\ell} x \, dx} = \frac{2}{\ell \left(1 - \frac{\sin 2\mu_k}{2\mu_k}\right)} \int_0^\ell \varphi(x) \sin \frac{\mu_k}{\ell} x \, dx$$

Podobne

$$u_t(x,t) = \sum_{k=1}^{\infty} \left(-a_k \sin \frac{\mu_k}{\ell} ct \cdot \frac{\mu_k}{\ell} c + b_k \cos \frac{\mu_k}{\ell} ct \cdot \frac{\mu_k}{\ell} c \right) \sin \frac{\mu_k}{\ell} x$$

$$u_t(x,0) = \sum_{k=1}^{\infty} b_k \frac{\mu_k}{\ell} c \sin \frac{\mu_k}{\ell} x = \psi(x)$$

$$b_k = \frac{2}{\mu_k c \left(1 - \frac{\sin 2\mu_k}{2\mu_k}\right)} \int_0^\ell \psi(x) \sin \frac{\mu_k}{\ell} x \, dx$$