

Teda

$$u_{\tau\tau} = 0$$

$$(u_{\tau})_{\tau} = 0$$

Integrácia podľa τ

$$u_{\tau} = h(\xi) \quad \text{závisí len od } \xi$$

Integrácia podľa ξ

$$u(\xi, \tau) = \int h(\xi) d\xi + \tilde{c}(\tau) = H(\xi) + \tilde{c}(\tau)$$

Pôvodné premenné

$$u(x, t) = H(x+ct) + \tilde{c}(x-ct)$$

Premenujeme

$$u(x, t) = f(x+ct) + g(x-ct)$$

Začiatková úloha.

$$u_{tt} + c^2 u_{xx} \quad x \in \mathbb{R} \quad t > 0$$

$$u(x, 0) = \varphi(x)$$

- začiatková poloha

$$u_t(x, 0) = \psi(x)$$

- začiatková rýchlosť

Riešenie.

$$u(x, t) = f(x+ct) + g(x-ct)$$

(riešenie rovnice)

Podmienka na polohu:

$$u(x, 0) = f(x) + g(x) = \varphi(x)$$

rýchlosť

$$u_t(x, 0) = f'(x) \cdot c - g'(x) \cdot c = \psi(x)$$

Riešime sústavu (pre neznáme f a g)

$$f'(x) + g'(x) = \varphi'(x)$$

$$c \cdot f'(x) - c g'(x) = \psi(x)$$

$$2c f'(x) = c \varphi'(x) + \psi(x)$$

$$2c g'(x) = c \varphi'(x) - \psi(x)$$

$$f(x) = \frac{1}{2c} \int c \varphi'(x) + \psi(x) dx = \frac{1}{2} \varphi(x) + \frac{1}{2c} \int_0^x \psi(\tilde{x}) d\tilde{x}$$

$$g(x) = \frac{1}{2c} \int c \varphi'(x) - \psi(x) dx = \frac{1}{2} \varphi(x) - \frac{1}{2c} \int_0^x \psi(\tilde{x}) d\tilde{x}$$

Riešenie

$$u(x, t) = \frac{1}{2} [\varphi(x+ct) + \varphi(x-ct)] + \frac{1}{2c} \left[\int_0^{x+ct} \psi(\tilde{x}) d\tilde{x} - \int_0^{x-ct} \psi(\tilde{x}) d\tilde{x} \right]$$

Problém 1. Najdite všeobecné řešení pro rovnici

$$u_{xx} + 2u_{xy} - 3u_{yy} = 0$$

Problém 2. $u_{xx} + 2u_{xy} + u_{yy} = 0$

Problém 3. $u_{xx} + 2u_{xy} + u_{yy} = 0$

$$u(x, 0) = \varphi(x)$$

$$u_y(x, 0) = \psi(x)$$

Problém 4. $u_{xx} + 2u_{xy} - 3u_{yy} = 0$

$$u(x, 0) = \varphi(x)$$

$$u_y(x, 0) = \psi(x)$$

$$u(x,t) = \frac{1}{2} [\varphi(x+ct) + \varphi(x-ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(\tilde{x}) d\tilde{x}$$

Príklad 1

$$u_{tt} = c^2 u_{xx}$$

$$u(x,0) = 0$$

$$u_t(x,0) = \sin x$$

$$u(x,t) = \frac{1}{2} \cdot 0 + \frac{1}{2c} \int_{x-ct}^{x+ct} \sin s ds = \frac{1}{2c} [-\cos s]_{x-ct}^{x+ct} = \frac{1}{2c} (\cos(x-ct) - \cos(x+ct))$$

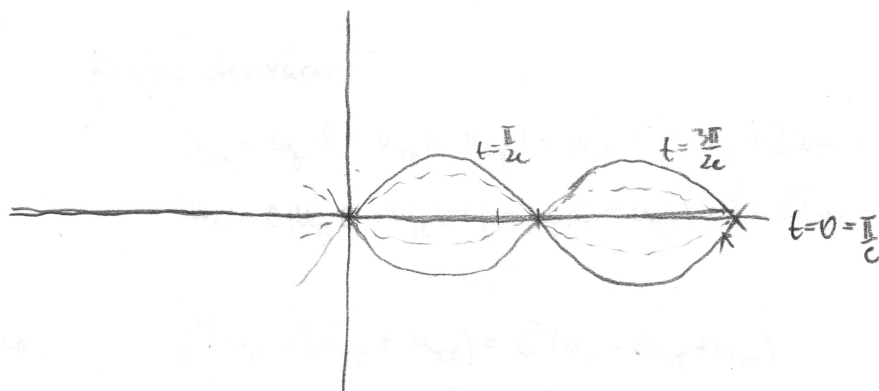
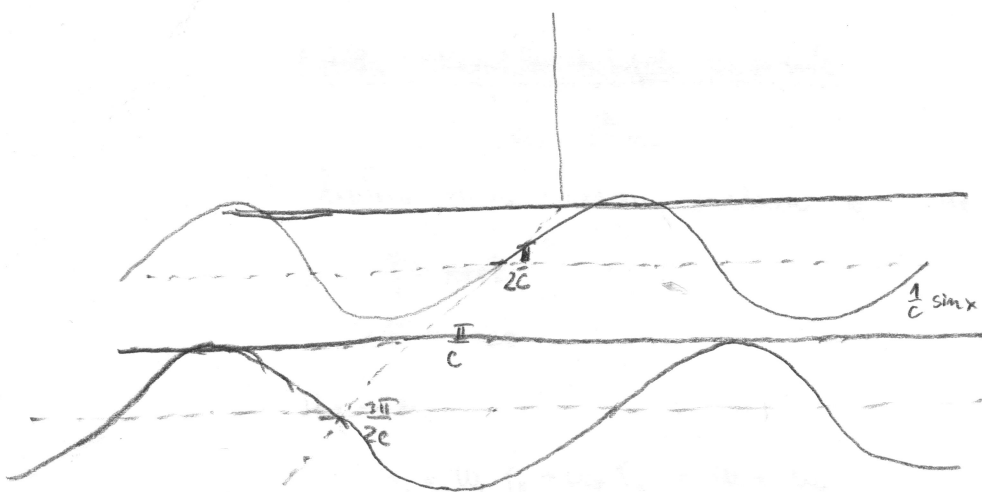
$$\left(\cos \alpha - \cos \beta = -2 \sin \frac{\alpha+\beta}{2} \sin \frac{\alpha-\beta}{2} \right)$$

$$u(x,t) = \frac{1}{2c} \cdot 2 \sin x \sin(-ct) = -\frac{1}{c} \sin x \sin ct$$

Kontrola podmienok:

$$u(x,0) = 0$$

$$u_t(x,0) = \frac{1}{c} \sin x \cdot c \cdot \cos 0 = \sin x$$



Príklad 2.

$$u_{tt} = c^2 u_{xx}$$

$$u(x, 0) = \sin x$$

$$u_t(x, 0) = 0$$

$$u(x, t) = \frac{1}{2} (\sin(x+ct) + \sin(x-ct))$$

$$\sin \alpha + \sin \beta = 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}$$

$$u(x, t) = \sin x \cos ct$$

Príklad 3

$$u_{tt} = c^2 u_{xx}$$

$$u(x, 0) = \sin x$$

$$u_t(x, 0) = \sin x$$

$$u(x, t) = \frac{1}{c} \sin x \sin ct + \sin x \cos ct = \sin x \left(\frac{1}{c} \sin ct + \cos ct \right)$$

$$u(x, t) = A \sin x \sin(ct - \phi)$$

$$A = \sqrt{1 + \frac{c^2}{c^2}}$$

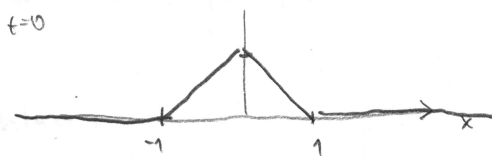
$$\phi = -\arctg \frac{1}{c}$$

Príklad 4

$$u_{tt} = c^2 u_{xx}$$

$$u(x, 0) = \begin{cases} 1-x & |x| \leq 1 \\ 0 & |x| > 1 \end{cases}$$

$$u_t(x, 0) = 0$$



Riešenie

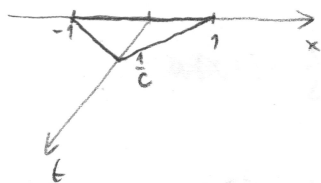
$$u(x, t) = \frac{1}{2} [\varphi(x+ct) + \varphi(x-ct)] =$$

$$\text{ak } |x+ct| \leq 1$$

$$\text{aj } |x-ct| \leq 1$$

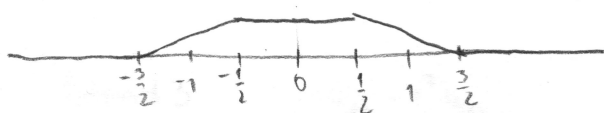
$$= \frac{1}{2} (1 - |x+ct| + 1 - |x-ct|) = 1 - \frac{|x+ct| + |x-ct|}{2}$$

keď je $|x+ct| \leq 1$ a $|x-ct| \leq 1$



Riešenie v čase $t = \frac{1}{2c}$

$$w(x, \frac{1}{2c}) = 1 - \frac{|x+\frac{1}{2}| + |x-\frac{1}{2}|}{2} \quad \text{pre } -\frac{1}{2} \leq x \leq \frac{1}{2}$$



$$\begin{aligned} |x+\frac{1}{2}| &\leq 1 \\ |x-\frac{1}{2}| &\leq 1 \end{aligned}$$



$$w(x, \frac{1}{2c}) = \frac{1 - |x - \frac{1}{2}|}{2} \quad \text{pre } \frac{1}{2} \leq x \leq \frac{3}{2}$$

$$\text{pre } \frac{1}{2} \leq x \leq \frac{3}{2}$$

$$w(x, \frac{1}{2c}) = 0 \quad \text{pre } \frac{3}{2} < x$$

$$\text{pre } \frac{3}{2} < x$$

Riešenie v čase $t = \frac{1}{c}$

$$w(x, \frac{1}{c}) = 0$$

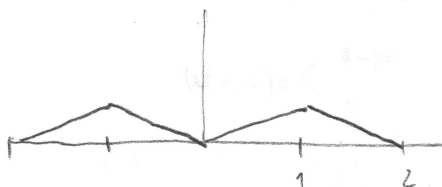
$$\text{pre } x=0$$

$$\text{pre } 0 \leq x \leq 2$$

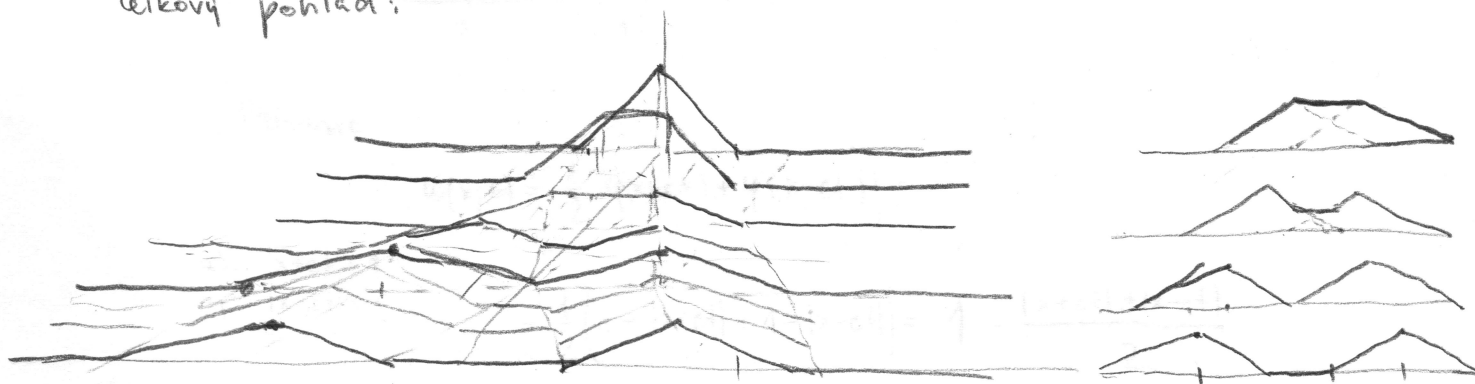
$$w(x, \frac{1}{c}) = \frac{1 - |x-1|}{2}$$

$$w(x, \frac{1}{c}) = 0 \quad \text{pre } 2 < x$$

$$\text{pre } 2 < x$$



Celkový pohľad:



Príklad 5

$$w_{tt} = c^2 u_{xx}$$

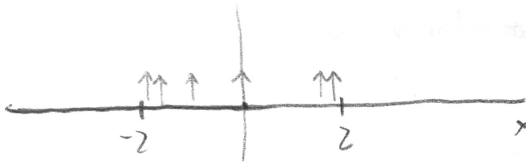
$$u(x, 0) = 0$$

$$u_t(x, 0) = \begin{cases} 1 & |x| \leq 2 \\ 0 & |x| > 2 \end{cases}$$

$$\text{pre } |x| \leq 2$$

$$|x| > 2$$

$$\psi(x)$$

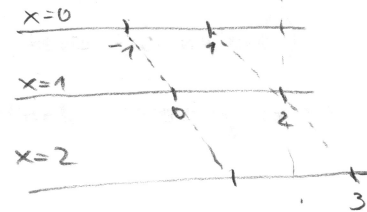
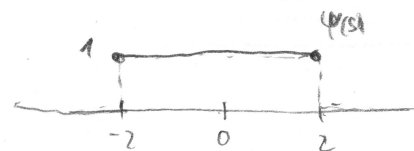
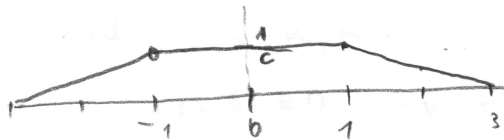


Riešenie

$$u(x, t) = \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) ds$$

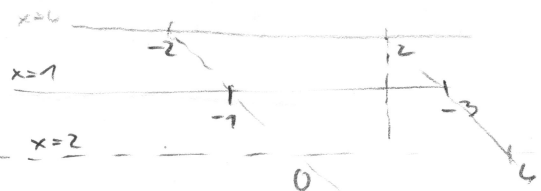
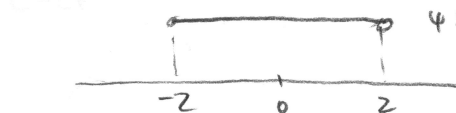
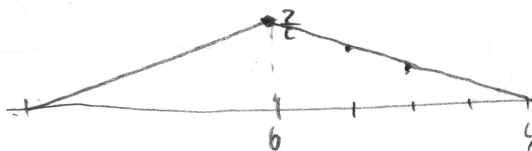
$$t = \frac{1}{c}$$

$$u(x, \frac{1}{c}) = \frac{1}{2c} \int_{x-1}^{x+1} \psi(s) ds$$



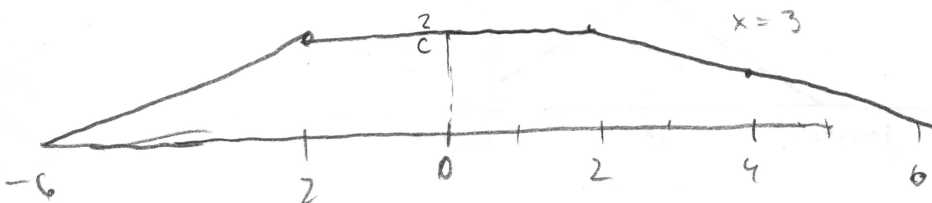
$$t = \frac{2}{c}$$

$$u(x, \frac{2}{c}) = \frac{1}{2c} \int_{x-2}^{x+2} \psi(s) ds$$



$$t = \frac{4}{c}$$

$$u(x, \frac{4}{c}) = \frac{1}{2c} \int_{x-4}^{x+4} \psi(s) ds$$



$$x = 0$$



$$x = 2$$

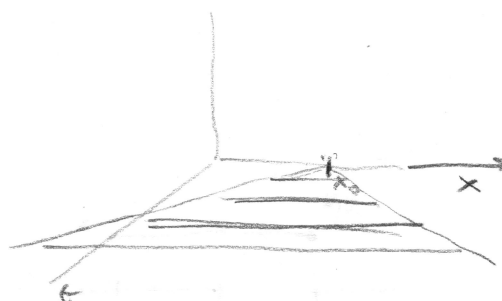
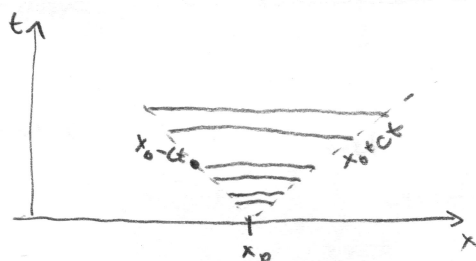


$$x = 3$$



Princíp kauzality

začiatková podmienka v čase $t=0$ daná v bode x_0 ovplyvní riešenie v čase t
 v prípade podmienky φ práve v bodoch x_0+ct a x_0-ct
 ψ v intervale (x_0-ct, x_0+ct)



To teda znamená, že ak sú zač. funkcie $\varphi(x)$ a $\psi(x)$ lokalizované
 na intervale $(-R, R)$ (a mimo neho sú nulové), tak
 riešenie $u(x, t) \equiv 0$ v tých bodoch roviny (x, t)
 kde je

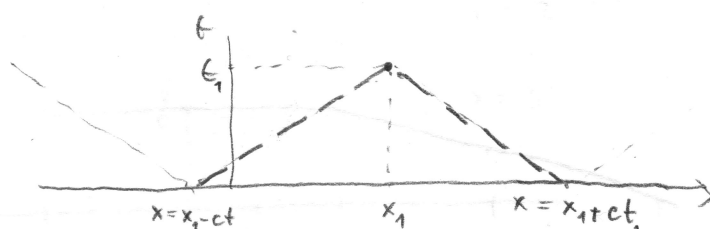
$$x > R+ct \quad \text{a} \quad x < -R-ct$$



Opačný pohľad: Oblasť histórie

Hodnota riešenia v bode x_1 a v čase t_1 : $u(x_1, t_1)$

je ovplyvnená v minulosti len hodnotami



Oblasť histórie:

Trojuholník

$$T = \{(x, t) ; x_1 - ct_1 \leq x \leq x_1 + ct_1\}$$

Vlnová rovnica so zdrojom. (nehomogénna)

Rovnica $u_{tt} - c^2 u_{xx} = f(x, t)$

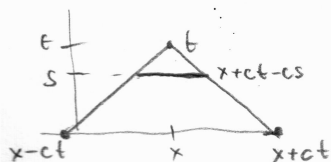
$u(x, 0) = \varphi(x) \quad u_t(x, 0) = \psi(x)$

Riesenie.

$$u(x, t) = \frac{1}{2} (\varphi(x+ct) + \varphi(x-ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(y) dy + u_p(x, t)$$

$$u_p(x, t) = \frac{1}{2c} \iint_{\Delta} f(y, s) dy ds = \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y, s) dy ds$$

Δ - oblasť histórie bodu (x, t)



Pr 1. $u_{tt} - c^2 u_{xx} = t \sin x$

$u(x, 0) = 1 + x^2 \quad u_t(x, 0) = 4 + x$

$$\begin{aligned} u(x, t) &= \frac{1}{2} [1 + (x+ct)^2 + 1 + (x-ct)^2] + \frac{1}{2c} \int_{x-ct}^{x+ct} (4+y) dy + \frac{1}{2c} \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} s \cdot \sin y dy ds = \\ &= 1 + x^2 + c^2 t^2 + \frac{1}{2c} \left[4y + \frac{y^2}{2} \right]_{x-ct}^{x+ct} + \frac{1}{2c} \int_0^t -s [\cos y]_{x-ct-s}^{x+ct-s} ds = \\ &= 1 + x^2 + c^2 t^2 + \frac{1}{2c} 8ct + \frac{1}{2c} 2xct + \frac{1}{2c} \int_0^t s \cos(x-ct-s) - \cos(x+ct-s) ds = \\ &= 1 + x^2 + c^2 t^2 + 4t + xt + \frac{1}{c} \int_0^t s \sin x \sin c(t-s) ds = \\ &= 1 + x^2 + c^2 t^2 + 4t + xt + \frac{\sin x}{c} \left(\left[s \frac{\cos c(t-s)}{c} \right]_0^t - \int_0^t \frac{\cos c(t-s)}{c} ds \right) = \\ &= 1 + x^2 + c^2 t^2 + 4t + xt + \frac{1}{c^2} \sin x \left(t + \left[\frac{\sin c(t-s)}{c} \right]_0^t \right) = \\ &= 1 + x^2 + c^2 t^2 + 4t + xt + \frac{1}{c^2} \sin x \cdot t - \frac{1}{3} \sin x \sin ct \end{aligned}$$