Príklad. Transportná rovnica s rozpadom

$$u_t + c u_x + \lambda u = 0 \qquad \lambda > 0,\ c > 0 \quad \text{konštanty}$$

$$u(x,0) = \frac{1}{x^2+1}$$

Použijeme súradnice

$$\xi = x - ct$$
$$\tau = t$$

$$u_t = u_\xi(-c) + u_\tau$$

$$u_x = u_\xi$$

$$-c\cancel{u_\xi} + u_\tau + c\cancel{u_\xi} + \lambda u = 0$$

$$u_\tau = -\lambda u$$

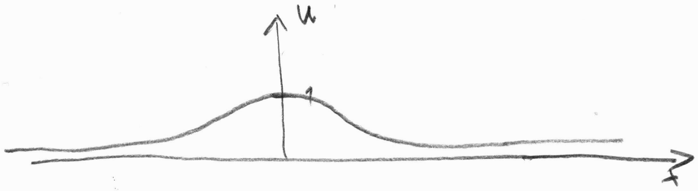
$$u(\xi,\tau) = \tilde{C}(\xi)\, e^{-\lambda\tau}$$

$$u(x,t) = \tilde{C}(x-ct)\, e^{-\lambda t}$$

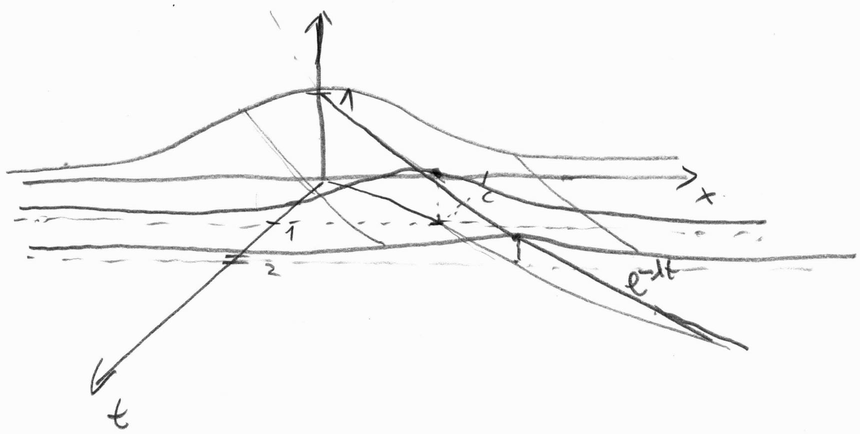
$$u(x,0) = \tilde{C}(x) = \frac{1}{x^2+1} \qquad \tilde{C}(w) = \frac{1}{w^2+1}$$

$$u(x,t) = \frac{1}{(x-ct)^2+1} \cdot e^{-\lambda t}$$

$u(x,0)$



$u(x,t)$



Príklad
$$u_t + c u_x + \lambda u = e^{-t}$$
$$u(x,0) = \frac{1}{x^2+1}$$

$$\xi = x - ct$$
$$\tau = t$$

$$u_t + \lambda u = e^{-\tau}$$

Homogénna rovnica

$$u_H = \hat{c}(\xi) e^{-\lambda\tau} \qquad u_H(x,t) = \tilde{c}(x-ct) \cdot e^{-\lambda t}$$
$$u_H(x,y) = \frac{1}{(x-ct)^2+1} \cdot e^{-\lambda t}$$

Nehomogénna rovnica, partikulárne riešenie

$$u_p(\xi,\tau) = \left(\int_0^\tau \frac{e^{-\tau}}{e^{-\lambda\tau}} d\tau \right) \cdot e^{-\lambda\tau} = \int_0^\tau e^{(\lambda-1)s} ds \cdot e^{-\lambda\tau} = \left[\frac{e^{(\lambda-1)s}}{\lambda-1} \right]_0^\tau e^{-\lambda\tau} =$$

$$= \frac{e^{(\lambda-1)\tau} - 1}{\lambda-1} \cdot e^{-\lambda\tau} = \frac{e^{-\tau} - e^{-\lambda\tau}}{\lambda-1}$$

$$u_p(x,t) = \frac{e^{-t} - e^{-\lambda t}}{\lambda-1}$$

$$\underline{\underline{u(x,t) = \left[\frac{1}{(x-ct)^2+1} - \frac{1}{\lambda-1} \right] e^{-\lambda t} + \frac{e^{-t}}{\lambda-1}}}$$

Problém
$$u_t + c u_x + \lambda u = f(t,x) = \begin{cases} e^{-t} & x \in \langle -1, 1 \rangle \\ 0 & x \notin \langle -1, 1 \rangle \end{cases}$$

Rovnice s nekonštantnými koeficientami

Metóda charakteristík

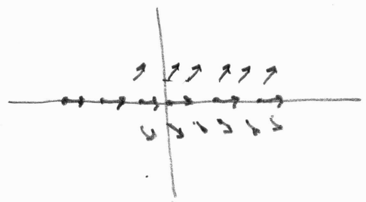
Príklad 1 Rovnica $u_x + y u_y = 0$

$a(x,y) = 1$

$b(x,y) = y$

definujme vektor $\vec{v} = (1, y)$ $\vec{v}(x,y)$ - nekonštantný

Vektorové pole



Prepísaná rovnica

$$(1,y)\cdot(u_x, u_y) = \vec{v}(x,y)\cdot \operatorname{grad} u = 0$$

znamená, že

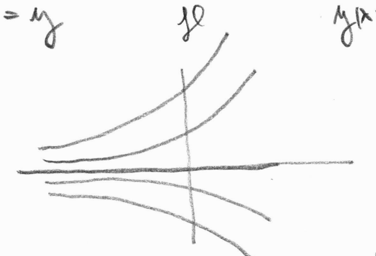
riešenie $u(x,y)$ je konštantné pozdĺž kriviek s dotykovým vektorom $\vec{v}(x,y)$

Rovnica pre tieto krivky je

$$y'(x) = \frac{dy}{dx} = \frac{y}{1}$$

a všeobecne $$y' = \frac{b(x,y)}{a(x,y)}$$

Riešenie rovnice $y' = y$ je $y(x) = c\cdot e^x$



Exponenciály $y = ce^x$ nazývame charakteristiky a každé riešenie musí byť konštantné pozdĺž charakteristiky (určenej voľbou konštanty c)

$u(x, y(x)) = u(x, ce^{x}) = F(c)$. $\quad$ F závisí len od c

$c = ye^{-x}$

teda $\quad u(x,y) = F(ye^{-x})$

Skúška. $\quad u_x + y u_y = F'(ye^{-x}) \cdot (-ye^{-x}) + y\, F'(ye^{-x}) \cdot e^{-x} = 0$

Poznámka. F nie je úplne ľubovoľná, tu predpokladáme, že má deriváciu.

Príklad 2 $\quad$ Úloha

$$u_x + y u_y = 0$$
$$u(0,y) = y+1.$$

Riešenie

$$u(x,y) = F(ye^{-x})$$
$$u(0,y) = F(y) = y+1 \qquad F(w) = w+1$$
$$\underline{\underline{u(x,y) = ye^{-x}+1}}$$

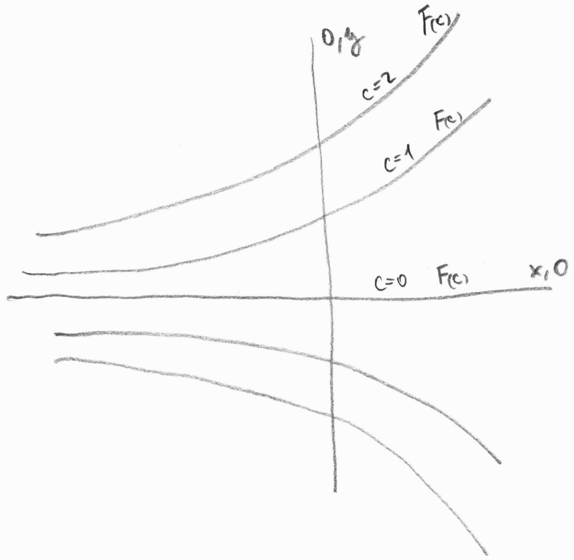
Príklad 3. $\quad$ Úloha

$$u_x + y u_y = 0$$
$$u(x,0) = x+1$$

$$u(x,y) = F(ye^{-x})$$
$$u(x,0) = F(0) = x+1$$

spor. $\quad$ Úloha nemá riešenie.

Prečo?



Príklad 4 $(1+x^2)u_x + u_y = 0$

Charakteristiky

$$y' = \frac{dy}{dx} = \frac{1}{1+x^2}$$

$$y(x) = \int \frac{1}{1+x^2}\,dx = \operatorname{arctg} x + c$$

$$c = y - \operatorname{arctg} x$$

$$u(x,y) = F(c) = F(y - \operatorname{arctg} x)$$

Príklad 5 $(1+x^2)u_x + u_y = 0$

$$u(0,y) = y^2$$

$$u(0,y) = F(y) = y^2 \qquad F(w) = w^2$$

$$u(x,y) = (y - \operatorname{arctg} x)^2$$

Príklad 6 $(1+x^2)u_x + u_y = 0$

$$u(x,0) = x$$

$$u(x,0) = F(-\operatorname{arctg} x) = x \qquad -\operatorname{arctg} x = w$$

$$x = \operatorname{tg}(-w)$$

$$F(w) = -\operatorname{tg} w$$

$$u(x,y) = -\operatorname{tg}(y - \operatorname{arctg} x) =$$

$$u(x,y) = -\frac{\operatorname{tg} y - x}{1 + x \operatorname{tg} y}$$

Def. obor riešenia?

Problém: Urobte skúšku a nakreslite D_u

Príklad 5. $u_x + 2xy^2 u_y = 0$

$$y'(x) = \frac{dy}{dx} = \frac{2xy^2}{1}$$

$$\frac{dy}{y^2} = 2x\,dx \qquad \int \frac{1}{y^2}\,dy = \int 2x\,dx$$

$$-\frac{1}{y} = x^2 + c \qquad x^2 + \frac{1}{y} = -c = \tilde{c}$$

$$u(x,y) = F\left(x^2 + \frac{1}{y}\right) \qquad \text{pre } y \neq 0$$

ak $y = 0$ $\quad u_x(x,0) = 0 \quad \Rightarrow \quad u(x,0) = \text{const.}$

Riešenie $\quad u(x,y) = \begin{cases} F\left(x^2 + \frac{1}{y}\right) & \text{pre } y \neq 0 \\ \text{const} & \text{pre } y = 0 \end{cases}$

je tzv. zovšeobecnené riešenie

Príklad 6. $u_x + 2xy^2 u_y = 0$

$u(0,y) = 1 + y$

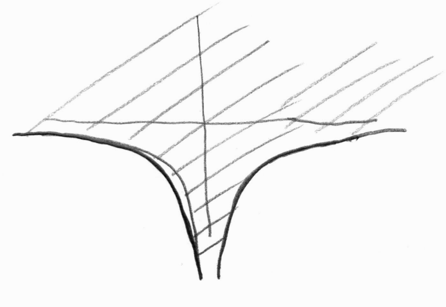
Riešenie pre $y \neq 0$ $\quad F\left(x^2 + \frac{1}{y}\right) = u(x,y)$

$$u(0,y) = F\left(\frac{1}{y}\right) = 1 + y \quad \Rightarrow \quad F(w) = 1 + \frac{1}{w} \qquad u(x,y) = 1 + \frac{1}{x^2 + \frac{1}{y}}$$

pre $y = 0$ $\quad u(0,0) = 1 \Rightarrow \text{const.} = 1$

$$u(x,y) = \begin{cases} 1 + \dfrac{1}{x^2 + \frac{1}{y}} & \text{pre } y \neq 0 \\ 1 & \text{pre } y = 0 \end{cases}$$

D_u :



Problém $\quad x u_x + y u_y = 0$

a, Zovšeobecnené riešenie

b, $\quad u(1,y) = y^2$

c, $\quad u(0,y) = y^2$