Hamiltonovské systémy

Skonštruujme konzervatívny systém k danému 1. integrálu
(Samozrejme riešení je nekonečne veľa...)

Dané je $H(x,y)$

Hľadáme systém $x' = f(x,y)$
$y' = g(x,y)$

Z podmienky na skalárny súčin

$$\operatorname{grad} H \cdot (f,g) = 0$$

je najjednoduchšie položiť

$$f(x,y) = \frac{\partial H}{\partial y} \quad \text{a} \quad g(x,y) = -\frac{\partial H}{\partial x}$$

Zrejme $\left(\frac{\partial H}{\partial x}, \frac{\partial H}{\partial y}\right) \cdot \left(\frac{\partial H}{\partial y}, -\frac{\partial H}{\partial x}\right) = 0$

Systém
$$x' = \frac{\partial H}{\partial y}(x,y)$$
$$y' = -\frac{\partial H}{\partial x}(x,y)$$
nazývame Hamiltonovský systém

a funkciu $H(x,y)$ (jej 1. integrál) nazývame Hamiltonián

Veta. V Hamiltonovskom systéme je divergencia poľa (f,g) rovná 0.

$$\operatorname{div}(f,g) = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = \frac{\partial^2 H}{\partial x \partial y} - \frac{\partial^2 H}{\partial y \partial x} = 0$$

Príklad 1
$x' = y$
$y' = -x$ Funkcia $H(x,y) = \frac{x^2}{2} + \frac{y^2}{2}$ je Hamiltonián systému

$\frac{\partial H}{\partial y} = y \qquad -\frac{\partial H}{\partial x} = -x$

Systém je konzervatívny aj Hamiltonovský.

Príklad 2. Systém $x' = -2xy$

$y' = +2xy - y$ je konzervatívny

Ale $\operatorname{div}(f,g) = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = -2y + 2x - 1 \neq 0$ nie je Hamiltonovský

Ako hľadať Hamiltonián pre daný systém?

Príklad 3. $x' = y$

$y' = -x + \alpha x^2$ $\alpha > 0$

$\frac{\partial H}{\partial y}(x,y) = y \quad \Rightarrow H(x,y) = \frac{y^2}{2} + c(x)$

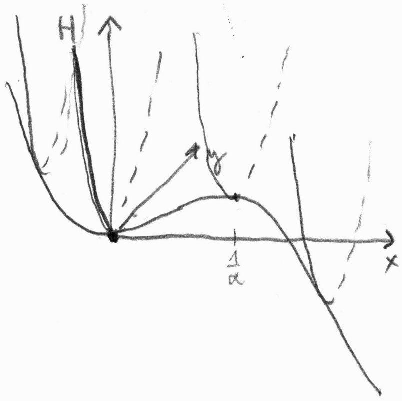
$-\frac{\partial H}{\partial x} = -c'(x) = -x + \alpha x^2 \qquad c'(x) = x - \alpha x^2$

$c(x) = \frac{x^2}{2} - \alpha \frac{x^3}{3} + c$

$H(x,y) = \frac{y^2}{2} + \frac{x^2}{2} - \alpha \frac{x^3}{3} + c$

Ak je systém Hamiltonovský, tak Hamiltonián je až na konštantu jediný.

Obrázok Hamiltoniánu

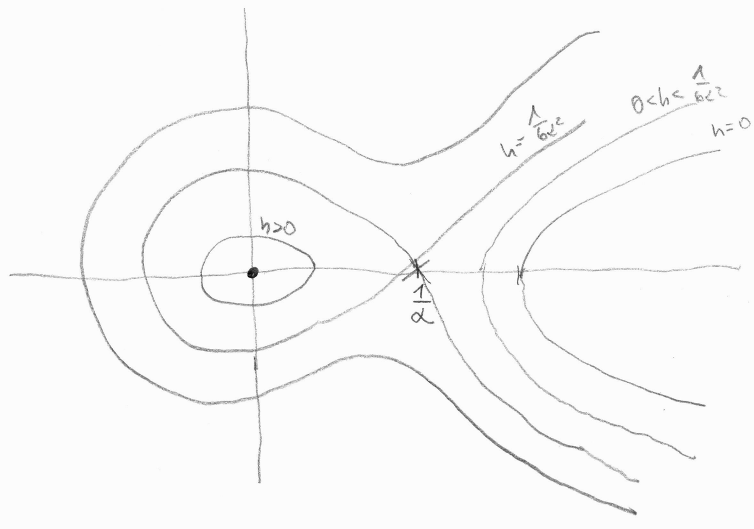


$[0,0]$ stred

$[\frac{1}{\alpha},0]$ sedlo

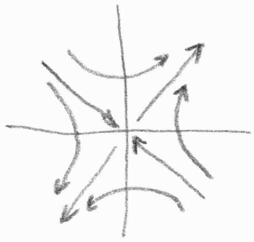
Fázový portrét

$\frac{1}{2\alpha^2} - \frac{1}{3\alpha^2} = \frac{1}{6\alpha^2}$



Linearizácia v sedle $DF(\vec{x}) = \begin{pmatrix} 0, & 1 \\ -1+2\alpha x, & 0 \end{pmatrix}$ $\quad DF(\frac{1}{\alpha},0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix}$

$\lambda^2 - 1 = 0 \qquad \lambda_1 = 1, \ \lambda_2 = -1 \qquad \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$



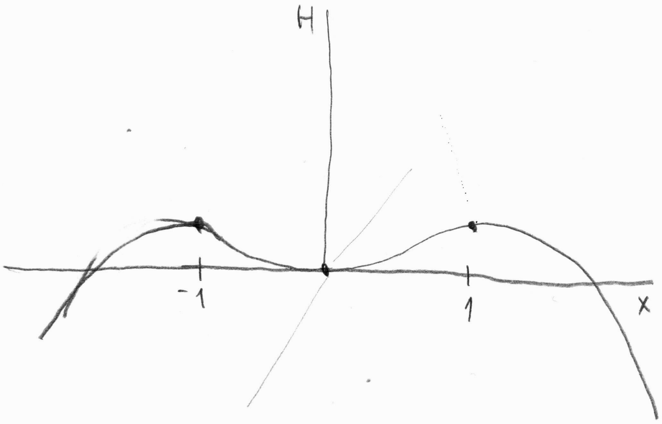
Príklad 4 Duffingov oscilátor

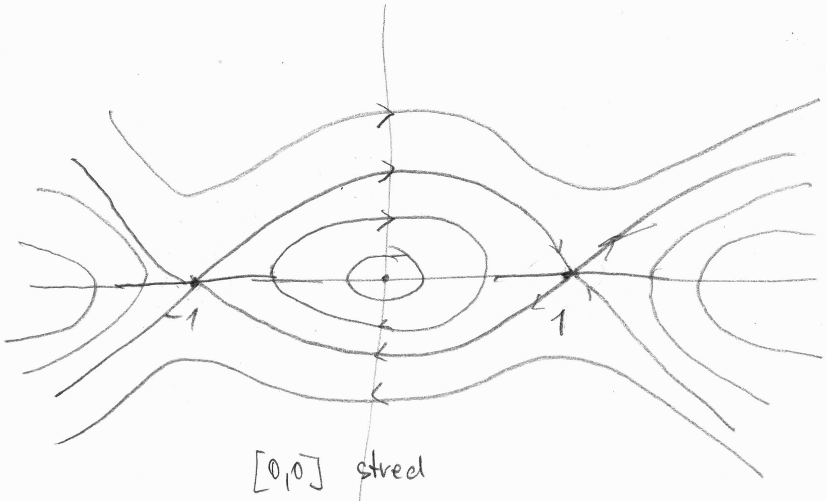
$$x'' + x - x^3 = 0 \qquad \begin{aligned} x' &= y \\ y' &= -x + x^3 \end{aligned}$$

Hamiltonián $\quad \frac{\partial H}{\partial y} = y \qquad H(x,y) = \frac{y^2}{2} + c(x)$

$$-\frac{\partial H}{\partial x} = -c'(x) = -x + x^3 \qquad c(x) = \int x - x^3 \, dx = \frac{x^2}{2} - \frac{x^4}{4} + c$$

$$H(x,y) = \frac{y^2}{2} + \frac{x^2}{2} - \frac{x^4}{4} + c$$





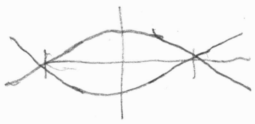
[0,0] stred

[1,0] [-1,0] sedlá

Linearizácia v sedlách

$$\begin{pmatrix} 0 & 1 \\ -1+3x^2 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \qquad \begin{vmatrix} -\lambda & 1 \\ 2 & -\lambda \end{vmatrix} = \lambda^2 - 2 = 0 \qquad \lambda_1 = \sqrt{2} \quad \lambda_2 = -\sqrt{2}$$

$$\begin{pmatrix} -\sqrt{2} & 1 \\ 2 & -\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} = 0$$



V konzervatívnych aj Hamiltonovských systémoch sú pevné body typu stred alebo sedlo. ...

Ak v Hamiltonovskom systéme urobíme linearizáciu, tak v pevnom bode c je

$$DF(c) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 H}{\partial x \partial y} & \frac{\partial^2 H}{(\partial y)^2} \\ -\frac{\partial^2 H}{(\partial x)^2} & -\frac{\partial^2 H}{\partial x \partial y} \end{pmatrix}$$

$$\det DF(c) = \frac{\partial^2 H}{(\partial y)^2}\frac{\partial^2 H}{(\partial x)^2} - \left(\frac{\partial^2 H}{\partial x \partial y}\right)^2 = H_{xx}H_{yy} - (H_{xy})^2 \qquad Tr(DF_c) = 0$$

Ak teda $H_{xx}H_{yy} - (H_{xy})^2 < 0$ tak c je sedlo

$H_{xx}H_{yy} - (H_{xy})^2 > 0$ c je stred

Podmienky súčasne znamenajú, že funkcia $H(x,y)$ má v sedle c sedlový bod, a v strede c lokálny extrém.

Príklad. 5 $x' = y$

$y' = x^3 + 2x^2 - x$

Pevné body $y = 0 \quad (x^2 + 2x - 1)x = 0 \qquad [0,0]$

$x_{1,2} = \frac{-2 \pm \sqrt{8}}{2} \qquad [-1+\sqrt{2}, 0], \ [-1-\sqrt{2}, 0]$

Hamiltonián $\frac{\partial H}{\partial y} = y \qquad H(x,y) = \frac{y^2}{2} + c(x)$

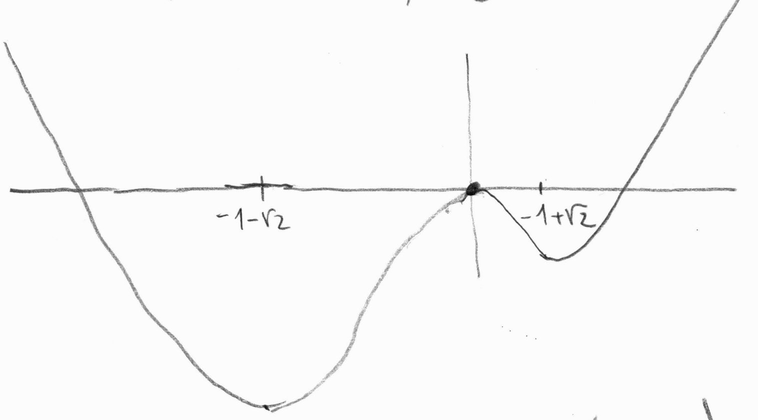
$$-\frac{\partial H}{\partial x} = -c'(x) = -x^3 - 2x^2 + x$$

$$c(x) = \frac{x^4}{4} + \frac{2}{3}x^3 - \frac{x^2}{2}$$

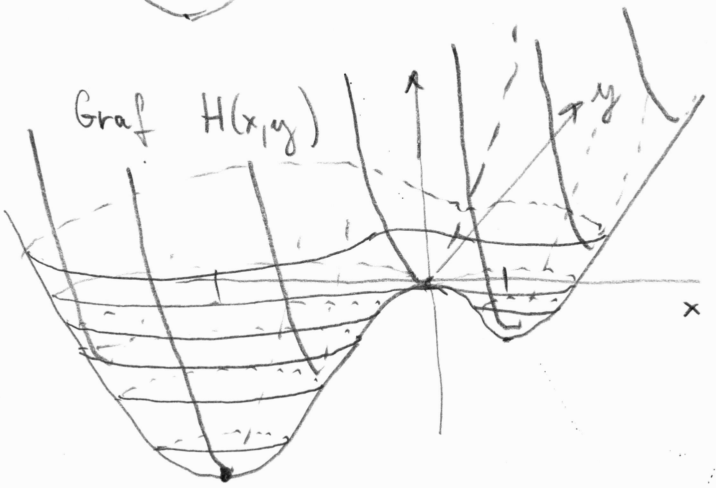
$$H(x,y) = \frac{y^2}{2} + \frac{x^4}{4} + \frac{2}{3}x^3 - \frac{x^2}{2} + c \qquad (\text{Volíme } c = 0)$$

$$H(x,y) = \frac{y^2}{2} + \frac{x^4}{4} + \frac{2}{3}x^3 - \frac{x^2}{2}$$

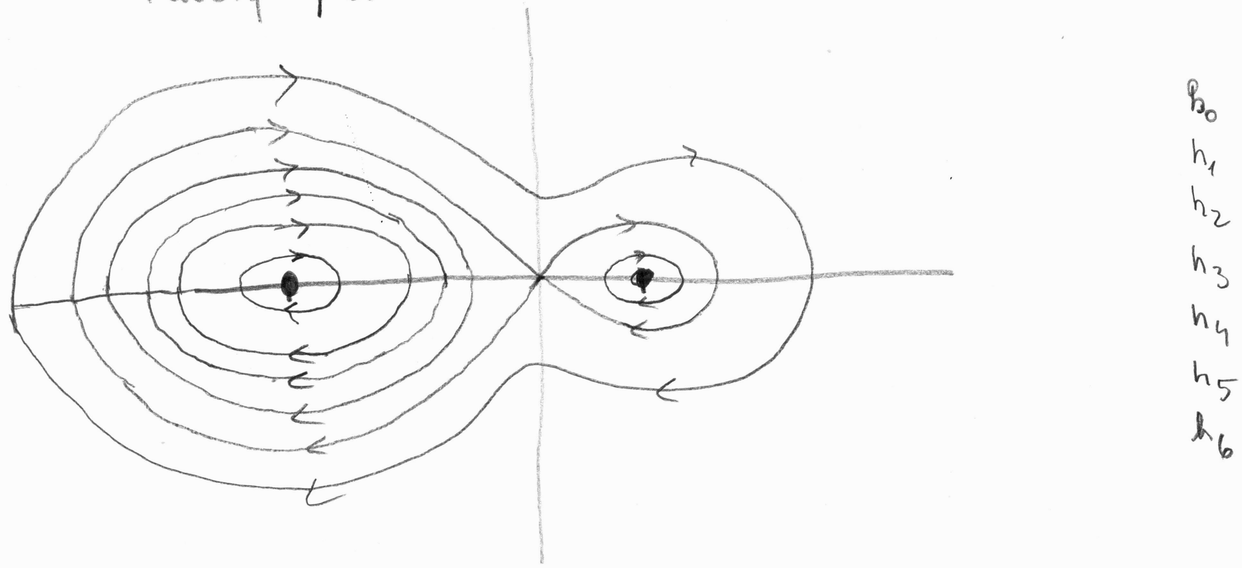
Graf $H(x,0) = \frac{x^4}{4} + \frac{2}{3}x^3 - \frac{x^2}{2}$



Graf $H(x,y)$



Fázový portrét



Konzervatívne systémy odvodené z rovnice

$$x'' + f(x) = 0$$

(mechanický systém bez trenia)

$$x' = y$$
$$y' = -f(x)$$

Pevné body $[x_i, 0]$ pre x_i korene $f(x)$.

Hamiltonián $H(x,y)$

$$\frac{\partial H}{\partial y} = y \qquad H(x,y) = \frac{y^2}{2} + c(x)$$

$$-\frac{\partial H}{\partial x} = -f(x) = -c'(x)$$

$$c(x) = \int f(x)\,dx = F(x)$$

$$H(x,y) = \frac{y^2}{2} + F(x) + c$$

Zovšeobecnená energia

Pre predstavu o fázovom portréte stačí kreslit' $H(x,0) = F(x)$

Minimá $F(x)$ sú tie korene x_i funkcie $f(x)$, kde má $H(x,y)$ v bode $[x_i, 0]$ minimum a teda sú to stredy

Maximá $F(x)$

$H(x,y)$ má v bode $[x_i, 0]$ sedlový bod a sú to sedlá.