1. Náhodná veličina je daná funkciou hustoty $f(x) = 2/(x+1)$ na intervale $[0, d]$, inde 0.

a) Určte hodnotu d.

b) Nájdite distribučnú funkciu $F(x)$ a kvantilovú funkciu.

c) Nakreslite grafy f, F. Vypočítajte $P\ (0.2 < x < 0.5)$ a znázornite ju v grafoch.

d) Nájdite medián, kvartily a 0.9-kvantil.

2. Náhodná veličina je daná distribučnou funkciou

$F(x) =$ / $(1-x^2)/2$ na $[-1, 0]$ (predtým 0)

\ $(1+x^2)/2$ na $[0, 1]$ (za tým 1)

a) Nájdite funkciu hustoty f a kvantilovú funkciu.

b) Nakreslite grafy f a F.

3. Náhodná veličina je daná distribučnou funkciou $F(x) = x^2/4$ na $[0, 2]$

a) Vypočítajte $E(X)$, $var(X)$

4. Náhodná veličina je daná funkciou hustoty

$f(x) =$ / $x+1$ na $[-1, 0]$

\ $1-x$ na $[0, 1]$, inde 0.

a) Nájdite distribučnú funkciu F a kvantilovú funkciu.

b) Nakreslite grafy f, F. Vypočítajte $P\ (-0.5 < x < 0.5)$ a znázornite ju v grafoch.

c) Nájdite medián, kvartily a 0.9-kvantil.

d) Vypočítajte $E(X)$, $var(X)$

5. Spoločná funkcia hustoty pre veličiny X, Y je daná

$f(x,y) =$ / k pre $0 < x < 1$ a $0 < y < a-x$

\ 0 inde.

a) Určte hodnotu k a nakreslite graf $f(x,y)$

b) Nájdite funkcie hustoty a distribučné funkcie veličín X a Y.

c) Vypočítajte $E(X)$, $E(Y)$, $var(X)$, $var(Y)$.

d) Vypočítajte $cov(X,Y)$ a korelačný koeficient.

6. Spoločná funkcia hustoty pre veličiny X, Y je daná

$f(x,y) =$ / $1-x+y$ pre $0 < x < 1$ a $0 < y < 1$

\ 0 inde.

a) Nakreslite 3D-graf $f(x,y)$

b) Nájdite funkcie hustoty a distribučné funkcie veličín X a Y.

c) Vypočítajte $E(X)$, $E(Y)$, $var(X)$, $var(Y)$.

d) Vypočítajte $cov(X,Y)$ a korelačný koeficient.

Riešenie úloh 1, 2

Pr-1 | Náhodná premenná X má hustotu $f(x) = \frac{2}{x+1}$ na $[0, d]$ | 5ta

a) Nájdite $F(x)$ a spresnite d!

$$F(d) = \int_0^d \frac{2}{x+1}\,dx = 2\left[\ln(x+1)\right]_0^d = 2\ln(d+1)$$

$$2\ln(d+1) \stackrel{!}{=} 1$$

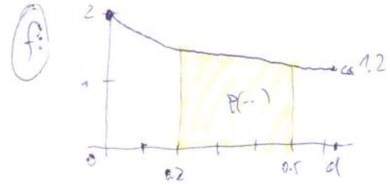
$$d+1 = e^{\frac{1}{2}}$$

$$d = e^{\frac{1}{2}} - 1 = \sqrt{e} - 1 = 0.6487212$$

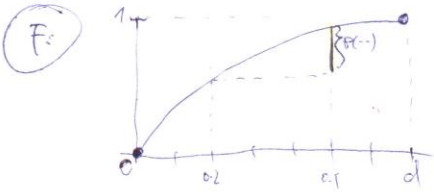
$$\boxed{F(x) = 2\ln(x+1)}$$

b) Načrtnite graf f, F, a $P(0.2 < x < 0.5)$:

f:



F:



c) Nájdite kvantilovú funkciu F^{-1}, nájdite medián, kvartily a 0.9 kvantil.

$$y = 2\ln(x+1)$$

$$e^{\frac{y}{2}} = x+1$$

$$x = e^{\frac{y}{2}} - 1$$

$$\boxed{F^{-1}(y) = e^{\frac{y}{2}} - 1}$$

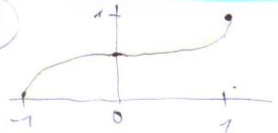
medián: $\tilde{x} = e^{0.25} - 1 \doteq 0.284$

kvartily: $Q_1 = e^{0.125} - 1 \doteq 0.13315$

$Q_3 = e^{0.375} - 1 \doteq 0.455$

$q_{0.9} = e^{0.45} - 1 = 0.5683122$

Pr-2 | F:



úryvky z paraboly:

$F(x) = \frac{1}{2}(1 - x^2)$ na $(-1, 0\rangle$, predtým 0

$F(x) = \frac{1}{2}(1 + x^2)$ na $(0, 1\rangle$, ďalej 1

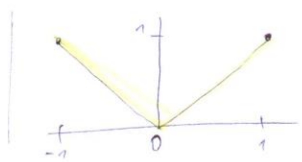
$F(x) = \frac{1}{2}(1 + x|x|)$ na $(-1, 1\rangle$

a) Nájdite f: (cez obr.)

$f = 0$ na $\mathbb{R} / (-1, 1\rangle$

$f = -x$ na $(-1, 0\rangle$

$f = x$ na $(0, 1\rangle$



b) Nájdite kvantil. fciu F^{-1}:

na $(-1, 0\rangle$: $y = \frac{1}{2}(1 - x^2) \Rightarrow y \in (0, 0.5)$!

$x^2 = 1 - 2y$

s mínusom $\rightarrow x = -\sqrt{1 - 2y}$

na $(0, 1\rangle$: $y = \frac{1}{2}(1 + x^2)$, $y \in (0, 1)$!

$x = +\sqrt{2y - 1}$

$$F^{-1} = \begin{cases} -\sqrt{1-2y} & \text{pre } y \in (0, 0.5) \\ \sqrt{2y-1} & \text{pre } y \in (0.5, 1\rangle \end{cases}$$

teda $\operatorname{sgn}(2y-1)\sqrt{|2y-1|}$

ďalej
to už
kvantily!

Riešenie úloh 3, 4

S4a

– toto je len alternatíva

(1) Je daná distribučná funkcia F náhodnej veličiny X:

$$\boxed{F(x) = \frac{x^2}{4} \text{ na } (0,2\rangle}$$

$F(x)=0$ pre $x \leq 0$
$F(x)=1$ pre $x > 2$
doplniť! (a:)

nakresliť grafy!

a) Nájdite hustotu náh. veličiny X:

Riešenie: $f(x) = F'(x) = \frac{x}{2}$ na $(0,2)$, inak 0

b) zistite $P(1 < X \leq 2)$

$$P(\cdot\cdot) = \int_1^2 f(x)\,dx = F(2) - F(1) = \frac{3}{4}$$

c) Vypočítajte $E(X)$, $\mathrm{var}(X)$:

$$E(x) = \int_0^2 x \cdot f(x)\,dx = \int_0^2 \frac{x^2}{2}\,dx = \frac{4}{3}$$

$$\mathrm{Var}(x) = E(x^2) - (E(x))^2 = \int_0^2 x^2 \cdot \frac{x}{2}\,dx - \frac{16}{9} = \frac{2}{9}$$

d) Nájdite kvantilovú funkciu F^{-1}:

$$y = \frac{x^2}{4} \rightarrow x = 2\sqrt{y}$$

- Medián: $y = 0.5$, $x = 2\sqrt{0.5} = \sqrt{2}$
- kvartily: $y = 0.25$, $x = 2\sqrt{0.25} = 1$, $y = 0.75$, $x = 2\sqrt{0.75} = \sqrt{3}$
- 0.9 kvantil: $y = 0.9$, $x = 2\sqrt{0.9} = 3\sqrt{\frac{2}{5}}$

e) Načrtnite graf – viď a), b)

(2) Je daná hustota f náh. veličiny X:

$$\boxed{f = \begin{matrix} x+1 & \text{na } [-1,0] \\ 1-x & \text{na } [0,1] \\ 0 & \text{inde} \end{matrix}}$$

grafy!

a) distrib. fcia F:

$$F = \int_{-1}^{x} f(x)\,dx:$$

Pre $x \leq -1$ $F(x) = 0$

Pre $x \in (-1,0\rangle$ $F(x) = \int_{-1}^{x} x+1\,dx = \frac{x^2+2x+1}{2}$, $F(0) = 0.5$

Pre $x \in (0,1\rangle$ $F(x) = 0.5 + \int_0^x 1-x\,dx = \frac{-x^2+2x+1}{2}$

Pre $x > 1$: $F(x) = 1$

$\frac{1}{2} + x - \frac{1}{2}|x|x$

b) $P(|x| < 0.5)$ – graficky + výpočet: $P(\cdot\cdot) = F(0.5) - F(-0.5) = (1 - \frac{1}{8}) - \frac{1}{8} = 0.75$

c) $E(X) = \int_{-1}^{1} x f(x) = \int_{-1}^{0} x(x+1) + \int_0^1 x(1-x) = 0$

$\mathrm{var}(x) = \int_{-1}^{1} x^2 f(x) = \int_{-1}^{0} x^2(x+1) + \int_0^1 x^2(1-x) = \frac{1}{6}$

d) kvantil F^{-1}:

$y = (x+1)^2/2$
$x = \sqrt{2y} - 1$ na $(0, \frac{1}{2})$

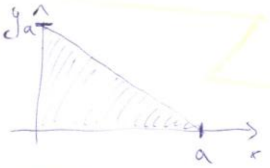
$y = 1 - \frac{(x-1)^2}{2}$
$x = 1 - \sqrt{2(1-y)}$ na $\langle \frac{1}{2}, 1\rangle$

$q_{0.5} = 0$
$q_{0.25} = \sqrt{0.5} - 1$
$q_{0.75} = 1 - \sqrt{0.5}$

Riešenie úlohy 5

Pr. 2 $\quad f(x,y) = k$ pre $0 < x < a,\ 0 < y < a - x$, inak 0 $\quad$ S4a

a) Náčrt: (plocha možnej daloši)



$k = ?$ $\quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f \, dx\,dy = \frac{a^2}{2} k \overset{!}{=} 1 \qquad k = \frac{2}{a^2}$

b) $f_x,\ f_y = ?$

$$f_x = \int_0^{a-x} k \, dy = \left[ky\right]_0^{a-x} = k(a-x) = \frac{2}{a^2}(a-x) \quad \text{pre } x \in [0,a]$$

$$f_y = \int_0^{a-y} k \, dx = \frac{2}{a^2}(a-y) \quad \text{pre } y \in [0,a]$$

c) $F(x),\ F(y) = ?$

$$F(x) = \int_0^x \frac{2}{a^2}(a-t)\,dt = \frac{2}{a^2}\left(at - \frac{t^2}{2}\right)_0^x = \frac{2}{a^2}\left(ax - \frac{x^2}{2}\right) = \frac{2ax - x^2}{a^2},\ [0,a]$$

$$F(y) = \frac{2ay - y^2}{a^2},\ \ldots$$

d) $E(x),\ E(y) = ?$

$$E(x) = \int_0^a x \cdot f_x = \int_0^a \frac{2}{a^2}(ax - x^2)\,dx = \frac{2}{a^2}\left[\frac{ax^2}{2} - \frac{x^3}{3}\right]_0^a = \frac{2}{a^2}\left[\frac{a^3}{2} - \frac{a^3}{3}\right] = \frac{2a}{6} = \boxed{\frac{a}{3}} \quad E(y)$$

e) $var(x) = var(y) = ?$

$$var(x) = \int_0^a x^2 f_x = \frac{2}{a^2}\int_0^a (ax^2 - x^3)\,dx = \frac{2}{a^2}\left[a\frac{x^3}{3} - \frac{x^4}{4}\right]_0^a = \frac{2}{a^2} \cdot \frac{a^4}{12} = \frac{a^2}{6} \qquad = var(y)$$

f) $cov(x,y) \Rightarrow$

$$cov(x,y) = \int_0^a\int_0^{a-x} xy\, f_{xy}\,dx\,dy - \frac{a^2}{9} = \int_0^a\int_0^{a-x} kxy\,dx\,dy - \frac{a^2}{9} = \int_0^a x\frac{(a-x)^2}{a^2}\,dx - \frac{a^2}{9} = \frac{a^2}{12} - \frac{a^2}{9} = -\frac{a^2}{36}$$

g) $$\rho(x,y) = \frac{-\frac{1}{36}a^2}{\sqrt{\frac{a^2}{6}\cdot\frac{a^2}{6}}} = -\frac{1}{6}$$

Riešenie úlohy 6

Náhodný vektor (X,Y): 5G a

$$F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f(u,v)\,du\,dv$$

~~Nezávislé ... ďalej~~

Príklad: 1

$$f(x,y) = \begin{cases} 1-x+y & \text{ak } (x,y) \in \langle 0,1\rangle^2 \\ 0 & \text{inak} \end{cases}$$ — nadol dole

a) $\boxed{f(x), f(y) = ?}$

je $f(x,y)$ dobre definovaná?

$$f_X = f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^1 1-x+y\,dy = \left[(1-x)y + \frac{y^2}{2}\right]_0^1 = (1-x)+\frac{1}{2} = \frac{3-2x}{2}$$

kontrola: $\int_0^1 \frac{3-2x}{2}\,dx = \frac{1}{2}\left[3x - x^2\right]_0^1 = 1$ ok

$$f_Y = f(y) = \int_0^1 1-x+y\,dx = \left[(1+y)x - \frac{x^2}{2}\right]_0^1 = \left(1+y-\frac{1}{2}\right) = \frac{1+2y}{2}$$

kontrola: $\int_0^1 \frac{1}{2}(1+2y)\,dy = \frac{1}{2}\left[y+y^2\right]_0^1 = 1$

b) $\boxed{F(x), F(y) = ? \quad F(x,y)}$

$$F(x) = \int_{-\infty}^{x} f_X(t)\,dt = \int_0^x \frac{3-2t}{2}\,dt = \frac{1}{2}\left[3t-t^2\right]_0^x = \frac{3x-x^2}{2}$$

$$F(y) = \int_0^y \frac{1+2t}{2}\,dt = \frac{1}{2}\left[t+t^2\right]_0^y = \frac{y+y^2}{2}$$

?

$$F(x,y) = \int_0^x\int_0^y 1-x+y\,dy\,dx = \int_0^x \left[(1-x)y + \frac{y^2}{2}\right]dx = \int_0^x \left(y+\frac{y^2}{2}\right) - yx\,dx = \left(y+\frac{y^2}{2}\right)x - \frac{yx^2}{2} = yx + \frac{yx(y-x)}{2} =$$

$$= yx\left[1+\frac{y-x}{2}\right]$$

c) $\boxed{E(X), E(Y) =}$

$$E(X) = \int_0^1 x f_X(x) = \int_0^1 x\frac{3-2x}{2}\,dx = \frac{1}{2}\left[\frac{3}{2}x^2 - \frac{2}{3}x^3\right]_0^1 = \frac{1}{2}\left(\frac{3}{2}-\frac{2}{3}\right) = \frac{1}{2}\cdot\frac{9-4}{6} = \frac{5}{12}$$

$$E(Y) = \int_0^1 y f_Y = \int_0^1 \frac{1}{2}(y+2y^2) = \frac{1}{2}\left(\frac{y^2}{2}+\frac{2}{3}y^3\right)_0^1 = \frac{1}{2}\left(\frac{1}{2}+\frac{2}{3}\right) = \frac{7}{12}$$

$\boxed{\mathrm{var}(X), \mathrm{var}(Y) = ?}$

$$\mathrm{var}(X) = \int_0^1 x^2 f_X\,dx - E^2(X) = \int_0^1 \frac{x^2}{2}(3-2x)\,dx - \frac{25}{144} = \frac{1}{2}\underbrace{\left[x^3 - \frac{1}{2}x^4\right]_0^1}_{0.25} - \frac{25}{144} = \frac{36-25}{144} = \frac{11}{144}$$

$$\mathrm{var}(Y) = \int_0^1 y^2 f_Y\,dy - E^2(Y) = \int_0^1 \frac{y^2}{2}+y^3\,dy - \frac{49}{144} = \left[\frac{y^3}{6}+\frac{y^4}{4}\right]_0^1 - \frac{49}{144} = \frac{5}{12} - \frac{49}{144} = \frac{11}{144}$$

d) $\boxed{\mathrm{cov}(X,Y) = ?}$

$$\mathrm{cov}(X,Y) = E(XY) - E(X)E(Y) = \int_0^1\int_0^1 xy(1-x+y)\,dx\,dy - \frac{35}{144} = \int_0^1\int_0^1 xy - x^2y + xy^2\,dy\,dx =$$

$$= \int_0^1\left[x\frac{y^2}{2} - x^2\frac{y^2}{2} + x\frac{y^3}{3}\right]_0^1 dx = \int_0^1\left[\frac{x-x^2}{2}+\frac{x}{3}\right]dx = \int_0^1 \frac{5}{6}x - \frac{x^2}{2}\,dx = \left[\frac{5}{12}x^2 - \frac{x^3}{6}\right]_0^1 \ldots = \frac{1}{4} - \frac{35}{144} = 0.00694 = \frac{1}{144}$$

TAKMER ⇒ SÚ NEZÁVISLÉ!

e) $\rho(X,Y) = ?$

$$\rho(X,Y) = \frac{\mathrm{cov}(X,Y)}{\sqrt{\mathrm{var}(X)\,\mathrm{var}(Y)}} = \frac{1}{144}\cdot\frac{1}{\frac{11}{144}} = \frac{1}{11} = 0.0909\ldots$$