Reťazové pravidlo

(1) Nech $t_0 \in R$ je vnútorným bodom oboru definície zloženej funkcie $F(t) = f(\overbrace{g(t)}^{x})$ $(g, f, F \subseteq R \times R)$. Nech existujú $g'(t_0)$ a $f'(g(t_0))$, potom

$$F'(t_0) = f'(g(t_0)) \cdot g'(t_0)$$

$$\left(\text{resp. } \left[\frac{dF(t)}{dt}\right]_{t=t_0} = \left[\frac{df(x)}{dx}\right]_{x=g(t_0)} \cdot \left[\frac{dg(t)}{dt}\right]_{t=t_0}\right)$$

$x = g(t)$ (mali sme v M1)

(2) Nech $t_0 \in R$ je vnútorným bodom oboru definície funkcie $F(t) = f(g_1(t), g_2(t), \dots, g_n(t))$ $(g_1, g_2, \dots, g_n, F \subseteq R \times R$, $f \subseteq R^n \times R)$

a nech $g_1, g_2, \dots, g_n$ sú diferencovateľné v bode t_0 (t.j. ex. $g_1'(t_0), \dots, g_n'(t_0)$) a f je diferencovateľná v bode $\bar{a} = (g_1(t_0), g_2(t_0), \dots, g_n(t_0))$. Potom existuje

$$F'(t_0) = \left[\frac{dF(t)}{dt}\right]_{t=t_0} = \left[\frac{\partial f(\bar{x})}{\partial x_1}\right]_{\bar{x}=\bar{a}} \cdot \left[\frac{dg_1(t)}{dt}\right]_{t=t_0} + \left[\frac{\partial f(\bar{x})}{\partial x_2}\right]_{\bar{x}=\bar{a}} \cdot \left[\frac{dg_2(t)}{dt}\right]_{t=t_0} + \dots + \left[\frac{\partial f(\bar{x})}{\partial x_n}\right]_{\bar{x}=\bar{a}} \left[\frac{dg_n(t)}{dt}\right]_{t=t_0}$$

(3) Nech $\bar{a} = (a_1, a_2, \dots, a_n) \in R^n$ je vnútorným bodom oboru definície zloženej funkcie

$$F(x_1, x_2, \dots, x_n) = f(\overbrace{g_1(x_1, \dots, x_n)}^{y_1}, \overbrace{g_2(x_1, \dots, x_n)}^{y_2}, \overbrace{g_m(x_1, \dots, x_n)}^{y_m})$$

$(g_1, g_2, \dots, g_m \subseteq R^n \times R$, $f \subseteq R^m \times R)$. Nech funkcie $g_1, \dots, g_m$ sú diferencovateľné v bode $\bar{a}$ a nech f je diferencovateľná v bode $\bar{b} = (g_1(\bar{a}), g_2(\bar{a}), \dots, g_m(\bar{a}))$. Potom existujú parciálne derivácie

$$\left[\frac{\partial F(\bar{x})}{\partial x_1}\right]_{\bar{x}=\bar{a}} = \left[\frac{\partial f(\bar{y})}{\partial y_1}\right]_{\bar{y}=\bar{b}} \cdot \left[\frac{\partial g_1(\bar{x})}{\partial x_1}\right]_{\bar{x}=\bar{a}} + \left[\frac{\partial f(\bar{y})}{\partial y_2}\right]_{\bar{y}=\bar{b}} \cdot \left[\frac{\partial g_2(\bar{x})}{\partial x_1}\right]_{\bar{x}=\bar{a}} + \dots + \left[\frac{\partial f(\bar{y})}{\partial y_m}\right]_{\bar{y}=\bar{b}} \cdot \left[\frac{\partial g_m(\bar{x})}{\partial x_1}\right]_{\bar{x}=\bar{a}}$$

$$\left[\frac{\partial F(\bar{x})}{\partial x_2}\right]_{\bar{x}=\bar{a}} = \left[\frac{\partial f(\bar{y})}{\partial y_1}\right]_{\bar{y}=\bar{b}} \cdot \left[\frac{\partial g_1(\bar{x})}{\partial x_2}\right]_{\bar{x}=\bar{a}} + \left[\frac{\partial f(\bar{y})}{\partial y_2}\right]_{\bar{y}=\bar{b}} \cdot \left[\frac{\partial g_2(\bar{x})}{\partial x_2}\right]_{\bar{x}=\bar{a}} + \dots + \left[\frac{\partial f(\bar{y})}{\partial y_m}\right]_{\bar{y}=\bar{b}} \cdot \left[\frac{\partial g_m(\bar{x})}{\partial x_2}\right]_{\bar{x}=\bar{a}}$$

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$$\left[\frac{\partial F(\bar{x})}{\partial x_n}\right]_{\bar{x}=\bar{a}} = \left[\frac{\partial f(\bar{y})}{\partial y_1}\right]_{\bar{x}=\bar{a}} \cdot \left[\frac{\partial g_1(\bar{x})}{\partial x_n}\right]_{\bar{x}=\bar{a}} + \left[\frac{\partial f(\bar{y})}{\partial y_2}\right]_{\bar{x}=\bar{a}} \cdot \left[\frac{\partial g_2(\bar{x})}{\partial x_n}\right]_{\bar{x}=\bar{a}} + \dots + \left[\frac{\partial f(\bar{y})}{\partial y_m}\right]_{\bar{y}=\bar{b}} \cdot \left[\frac{\partial g_m(\bar{x})}{\partial x_n}\right]_{\bar{x}=\bar{a}}$$

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n-rovností hore môžeme zapísať:

(3)
$$\left[\frac{\partial F(\bar{x})}{\partial x_k}\right]_{\bar{x}=\bar{a}} = \left[\frac{\partial f(\bar{y})}{\partial y_1}\right]_{\bar{y}=\bar{b}} \cdot \left[\frac{\partial g_1(\bar{x})}{\partial x_k}\right]_{\bar{x}=\bar{a}} + \left[\frac{\partial f(\bar{y})}{\partial y_2}\right]_{\bar{y}=\bar{b}} \cdot \left[\frac{\partial g_2(\bar{x})}{\partial x_k}\right]_{\bar{x}=\bar{a}} + \dots + \left[\frac{\partial f(\bar{y})}{\partial y_m}\right]_{\bar{y}=\bar{b}} \cdot \left[\frac{\partial g_m(\bar{x})}{\partial x_k}\right]_{\bar{x}=\bar{a}}$$

pre $k = 1, 2, \dots, n$ (teda mení sa len x_k)!!

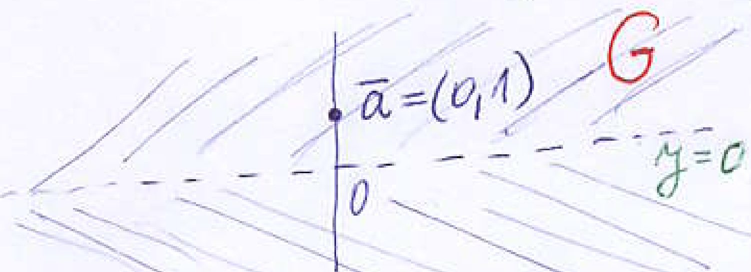
$(y_1 = g_1(x_1, \dots, x_n),\ y_2 = g_2(x_1, \dots, x_n), \dots, y_m = g_m(x_1, \dots, x_n))$

Nech $F(x,y) = f\left(e^{2x+y}, \frac{3x}{y}, x\right)$ a nech $f \subseteq R^3 \times R$ je diferencovateľná v každom $\bar{b} \in R^3$. Nech $\bar{a} = (0,1) \in R^2$. Najdite $\left[\frac{\partial F(x,y)}{\partial x}\right]_{\substack{x=0\\y=1}}$, $\left[\frac{\partial F(x,y)}{\partial y}\right]_{\substack{x=0\\y=1}}$.

(a) funkcie

$$\left.\begin{aligned} u_1(x,y) &= e^{2x+y} \\ u_2(x,y) &= \frac{3x}{y} \\ u_3(x,y) &= x \end{aligned}\right\}$$

sú diferencovateľné v každom bode (x,y) z množiny $G = R^2 \setminus \underbrace{\{(x,y) \in R^2 \mid y = 0\}}_{os\ x}$



$\bar{a} \in G$

prekože G je otvorená podmnožina R^2 a funkcie u_1, u_2, u_3 majú na nej (t.j. v každom jej bode) spojité 1-vé parciálne derivácie

$$\frac{\partial u_1(x,y)}{\partial x} = 2 \cdot e^{2x+y}, \quad \frac{\partial u_1(x,y)}{\partial y} = e^{2x+y},$$

$$\frac{\partial u_2(x,y)}{\partial x} = \frac{3}{y}, \quad \frac{\partial u_2(x,y)}{\partial y} = -\frac{3x}{y^2},$$

$$\frac{\partial u_3(x,y)}{\partial x} = 1, \quad \frac{\partial u_3(x,y)}{\partial y} = 0.$$

(b) f je diferencovateľná v každom $(u_1,u_2,u_3) \in R^3$, Teda podľa reťazového pravidla (3) : v bode $\bar{a} = (0,1)$ a teda $\bar{b} = (u_1(0,1), u_2(0,1), u_3(0,1)) = (e,0,0)$

je:

$$\boxed{\left[\frac{\partial F(x,y)}{\partial x}\right]_{\substack{x=0\\y=1}}} = \left[\frac{\partial f(u_1,u_2,u_3)}{\partial u_1}\right]_{\substack{u_1=e\\u_2=0\\u_3=0}} \cdot \left[\frac{\partial u_1(x,y)}{\partial x}\right]_{\substack{x=0\\y=1}} +$$

pokračovanie

$$+ \left[\frac{\partial f(u_1,u_2,u_3)}{\partial u_2}\right]_{\substack{u_1=e\\u_2=0\\u_3=0}} \cdot \left[\frac{\partial u_2(x,y)}{\partial x}\right]_{\substack{x=0\\y=1}} + \left[\frac{\partial f(u_1,u_2,u_3)}{\partial u_3}\right]_{\substack{u_1=e\\u_2=0\\u_3=0}} \cdot \left[\frac{\partial u_3(x,y)}{\partial x}\right]_{\substack{x=0\\y=1}} =$$

$$= \left[\frac{\partial f(u_1,u_2,u_3)}{\partial u_1}\right]_{\substack{u_1=e\\u_2=0\\u_3=0}} \cdot 2e + \left[\frac{\partial f(u_1,u_2,u_3)}{\partial u_2}\right]_{\substack{u_1=e\\u_2=0\\u_3=0}} \cdot 3 + \left[\frac{\partial f(u_1,u_2,u_3)}{\partial u_3}\right]_{\substack{u_1=e\\u_2=0\\u_3=1}} \cdot 1$$

$$\boxed{\left[\frac{\partial F(x,y)}{\partial y}\right]_{\substack{x=0\\y=1}}} = \left[\frac{\partial f(u_1,u_2,u_3)}{\partial u_1}\right]_{\substack{u_1=e\\u_2=0\\u_3=0}} \cdot \left[\frac{\partial u_1(x,y)}{\partial y}\right]_{\substack{x=0\\y=1}} + \left[\frac{\partial f(u_1,u_2,u_3)}{\partial u_2}\right]_{\substack{u_1=e\\u_2=0\\u_3=0}} \cdot \left[\frac{\partial u_2(x,y)}{\partial y}\right]_{\substack{x=0\\y=1}}$$

$$+ \left[\frac{\partial f(u_1,u_2,u_3)}{\partial u_3}\right]_{\substack{u_1=e\\u_2=0\\u_3=0}} \cdot \left[\frac{\partial u_3(x,y)}{\partial y}\right]_{\substack{x=0\\y=1}} =$$

$$= \left[\frac{\partial f(u_1,u_2,u_3)}{\partial u_1}\right]_{\substack{u_1=e\\u_2=0\\u_3=0}} \cdot e + \left[\frac{\partial f(u_1,u_2,u_3)}{\partial u_2}\right]_{\substack{u_1=e\\u_2=0\\u_3=0}} \cdot 0 + \left[\frac{\partial f(u_1,u_2,u_3)}{\partial u_3}\right]_{\substack{u_1=e\\u_2=0\\u_3=1}} \cdot 0$$

$$= \left[\frac{\partial f(u_1,u_2,u_3)}{\partial u_1}\right]_{\substack{u_1=e\\u_2=0\\u_3=0}} \cdot e$$

Poznámka. Ak funkcie v (3) sú explicitne dané je výpočet jednoduchší:

$$F(x,y) = x^2 \ln(x^2+y^2+1) \quad \left\{ \begin{array}{l} F(x,y) = f(u_1(x,y), u_2(x,y)) \\ f(u_1,u_2) = u_1 \cdot u_2 \\ u_1(x,y) = x^2 \\ u_2(x,y) = \ln(x^2+y^2+1) \end{array} \right.$$

$$\frac{\partial F(x,y)}{\partial x} = 2x \ln(x^2+y^2+1) + x^2 \frac{2x}{x^2+y^2+1}$$

$$\frac{\partial F(x,y)}{\partial y} = x^2 \frac{2y}{x^2+y^2+1}$$

existujú obe v hocikom $(x,y) \in R^2$