Diferencovateľnosť a diferenciál funkcie $f \subseteq R^n \times R$ v bode $\bar{a}$ (existuje $O_\delta(\bar{a}) \subseteq D(f)$).

zopakujme si: Ak bod $\bar{a} \in R^n$ je vnútorným bodom $D(f)$ funkcie $f \subseteq R^n \times R$, tak:

(1) f nazývame **diferencovateľnou v $\bar{a}$** ak

existujú $\left[\frac{\partial f(\bar{x})}{\partial x_k}\right]_{\bar{x}=\bar{a}}$, $k = 1, 2, \dots, n$

funkcie $\varepsilon_k \subseteq R^n \times R$ také že $\lim\limits_{\bar{x} \to \bar{a}} \varepsilon_k(\bar{x}) = \varepsilon_k(\bar{a}) = 0$ pre $k = 1, 2, \dots, n$

také že platí:

pre všetky $\bar{x} \in O_\delta(\bar{a})$:

$$f(\bar{x}) - f(\bar{a}) = \sum_{k=1}^{n} \left[\frac{\partial f(\bar{x})}{\partial x_k}\right]_{\bar{x}=\bar{a}} (x_k - a_k) + \sum_{k=1}^{n} \varepsilon_k(\bar{x})(x_k - a_k)$$

(2) Funkciu

$$Df_{\bar{a}}(\bar{x}) = \sum_{k=1}^{n} \left[\frac{\partial f(\bar{x})}{\partial x_k}\right]_{\bar{x}=\bar{a}} (x_k - a_k)$$

nazývame **diferenciálom funkcie f v bode $\bar{a}$.**

Poznámka: Ak $f \subseteq R^n \times R$ je diferencovateľná v bode $\bar{a} = (a_1, a_2, \dots, a_n)$ a pre dané $(h_1, h_2, \dots, h_k) = \bar{h}$ je $x_k = a_k + h_k$, $k = 1, 2, \dots, n$ (t.j. $x_k - a_k = h_k$) potom:

$$\boxed{f(a_1 + h_1, \dots, a_n + h_n) - f(a_1, \dots, a_n)} \doteq \sum_{k=1}^{n} \left[\frac{\partial f(\bar{x})}{\partial x_k}\right]_{\bar{x}=\bar{a}} \cdot h_k = \underbrace{Df_{\bar{a}}(\bar{a} + \bar{h})}$$

je približne rovné číslu

- $f(x,y) = \sqrt{x^2+y^2}$; $\bar{a} = (3,4)$

$$\frac{\partial f(x,y)}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}, \quad \frac{\partial f(x,y)}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

$$f(\bar{a}) = f(3,4) = \sqrt{9+16} = \sqrt{25} = \underline{5}$$

$$\left[\frac{\partial f(x,y)}{\partial x}\right]_{\bar{x}=\bar{a}} = \left[\frac{x}{\sqrt{x^2+y^2}}\right]_{\substack{x=3\\y=4}} = \frac{3}{5} ; \left[\frac{\partial f(x,y)}{\partial y}\right]_{\substack{x=3\\y=4}} = \frac{4}{5}$$

$$Df_{\bar{a}}(\bar{x}) = \frac{3}{5}(x-3) + \frac{4}{5}(y-4)$$

$\left[\frac{\partial f(\bar{x})}{\partial x}\right]_{\bar{x}=\bar{a}}$ (*)

pre $\bar{h} = (h_1, h_2)$ je $Df_{\bar{a}}(\bar{a}+\bar{h}) = \frac{3}{5} \cdot h_1 + \frac{4}{5} \cdot h_2$

a teda platí: $f(3+h_1, 4+h_2) - f(3,4) \doteq \frac{3}{5} \cdot h_1 + \frac{4}{5} \cdot h_2$

(podľa poznámky o $Df_{\bar{a}}(\bar{a}+\bar{h})$ je) (**)

$$\Downarrow$$

$$f(3+h_1, 4+h_2) \doteq \underbrace{f(3,4)}_{=5} + \frac{3}{5}h_1 + \frac{4}{5}h_2$$

teda napríklad ; z (*) resp (**):

$$\underline{f(3,1 ; 3,9)} = f(3+0,1 ; 4-0,1) \doteq 5 + \frac{3}{5} \cdot \underbrace{\frac{1}{10}}_{=h_1} + \frac{4}{5}\underbrace{\left(-\frac{1}{10}\right)}_{=h_2}$$

$$= 5 + \frac{3}{50} - \frac{4}{50} = 5 - \frac{1}{50} = 4,98$$

(priamym výpočtom je $f(3,1 ; 3,9) = \sqrt{24,82} = 4,981\ldots$)

$$\underline{f(3,012 ; 3,997)} \doteq 5 + \frac{3}{5} \cdot 0,012 + \frac{4}{5}(-0,003) =$$

$$\underline{f(3+0,12 ; 4-0,003)} \doteq 5,0048$$

platí:

(1) f je diferencov. v $\bar{a} \Leftrightarrow \lim\limits_{\bar{x}\to\bar{a}} \dfrac{f(\bar{x})-f(\bar{a})-Df_{\bar{a}}(\bar{x})}{\|\bar{x}-\bar{a}\|}=0$

(2) f je diferencov. v $\bar{a} \Rightarrow f$ je spojitá v $\bar{a}$

$\not\Leftarrow$

(3) existujú $\left[\dfrac{\partial f(\bar{x})}{\partial x_k}\right]_{\bar{x}=\bar{a}}$ pre $k=1,\dots,n$ $\not\Rightarrow$ f je diferencov. v $\bar{a}$

(4) existujú $\dfrac{\partial f(\bar{x})}{\partial x_k}$ na $O_\delta(\bar{a})$ a sú spojité v $\bar{a} \Rightarrow f$ je diferencov. v $\bar{a}$

- $f(x,y) = \begin{cases} \dfrac{\sin 6xy}{y} & \text{pre } y\neq 0 \\ 6x & \text{pre } y=0 \text{ (t.j. v bodoch } (x,0)) \end{cases}$

nech $\bar{a}=(3,0)$

$f(3,0)=6.3=18$

$\lim\limits_{\substack{x\to 3\\ y\to 0}} f(x,y) = \lim\limits_{\substack{x\to 3\\ y\to 0}} \overset{\to 18}{6x}\,\overset{\to 1}{\dfrac{\sin 6xy}{6xy}} = \underset{\substack{\| \\ f(3,0)}}{18}$

f je spojitá v $\bar{a}$

pretože: $\lim\limits_{\substack{x\to 3\\ y\to 0}} 6x = 6.3=18$ & $\lim\limits_{\substack{x\to 3\\ y\to 0}} \dfrac{\sin(6xy)}{6xy}=1$

pretože: $\lim\limits_{z\to 0}\dfrac{\sin z}{z} \underset{\text{L'Hospital}}{=} \lim\limits_{z\to 0}\cos z=\cos 0=1 \Rightarrow$

$\Rightarrow \lim\limits_{\substack{x\to 3\\ y\to 0}} \dfrac{\sin(\overset{z\to 0}{6xy})}{\underset{z\to 0}{6xy}} = \lim\limits_{z\to 0}\dfrac{\sin z}{z}=1$

$\left[\dfrac{\partial f(x,y)}{\partial y}\right]_{\substack{x=3\\ y=0}} = \lim\limits_{y\to 0}\dfrac{f(3,y)-f(3,0)}{y-0} = \lim\limits_{y\to 0}\dfrac{\dfrac{\sin 18y}{y}-6.3}{y} =$

$= \lim\limits_{y\to 0}\dfrac{\sin(18y)-18y}{y^2} = \lim\limits_{y\to 0}\dfrac{18\cos(18y)-18}{2y} =$

$$\underset{\text{L'Hospital}}{=} \lim_{y\to 0} \frac{18^2(-\sin 18y)}{2} = \underline{0}$$

$$\left[\frac{\partial f(x,y)}{\partial x}\right]_{\substack{x=3\\y=0}} = \lim_{x\to 3} \frac{f(x,0)-f(3,0)}{x-3} = \lim_{x\to 3}\frac{6x-18}{x-3} =$$

$$= \lim_{x\to 3} 6\,\frac{x-3}{x-3} = \lim_{x\to 3} 6 = \underline{6}$$

$$Df_{\bar a}(\bar x) = \left[\frac{\partial f(x,y)}{\partial x}\right]_{\substack{x=3\\y=0}}(x-3) + \left[\frac{\partial f(x,y)}{\partial y}\right]_{\substack{x=3\\y=0}}(y-0) =$$

$$\underline{\bar a = (3,0)} \quad = 6(x-3) + 0\cdot(y-0) = \underline{6x-18}$$

$$\lim_{\substack{x\to 3\\y\to 0}} \frac{f(x,y)-f(3,0)-Df_{(3,0)}(x,y)}{\|(x,y)-(3,0)\|} = \lim_{\substack{x\to 3\\y\to 0}} \frac{\frac{\sin 6xy}{y} - \cancel{18} + 6x + \cancel{18}}{\sqrt{(x-3)^2+(y-0)^2}}$$

$$= \lim_{\substack{x\to 3\\y\to 0}} \frac{\sin 6xy - 6xy}{y\sqrt{(x-3)^2+y^2}} = \lim_{\substack{x\to 3\\y\to 0}} \frac{\sin 6xy - 6xy}{(6xy)^2}(6x)^2\frac{y^2}{y\sqrt{(x-3)^2+y^2}} =$$

$= 0$, pretože $\dfrac{y^2}{y\sqrt{(x-3)^2+y^2}} \in \langle -1, 1\rangle$ (ohraničená)

a $\displaystyle\lim_{\substack{x\to 3\\y\to 0}} \frac{\sin(6xy)-6xy}{(6xy)^2} = 0$ pretože (ak $xy = u$, tak $u \to 0$)

$$\lim_{u\to 0}\frac{\sin u - u}{u^2} \underset{\text{L'Hospital}}{=} \lim_{u\to 0}\frac{\cos u - 1}{2u} \underset{\text{L'Hospital}}{=} \lim_{u\to 0}\frac{\sin u}{2} = 0$$

a teda dostávame $\displaystyle\lim_{\substack{x\to 3\\y\to 0}} \frac{\sin 6xy - 6xy}{(6xy)^2}(6x)^2 = 0\cdot 18^2 = \underline{\underline{0}}$

Teda f je diferencovateľná v bode $(3,0)$.

Ukážme, že pre $n=1$ funkcia $f \subseteq R \times R$ je diferencovateľná v bode $a \in R$ takom že exist. $O_\delta(a) \subseteq D(f)$ práve ak existuje $f'(a)$.

existuje $f'(a) = \lim\limits_{x \to a} \dfrac{f(x)-f(a)}{x-a} \Longrightarrow$

$$\Longrightarrow \quad \left.\begin{array}{l} \varepsilon(x) = \begin{cases} \dfrac{f(x)-f(a)}{x-a} - f'(a), & \text{pre } x \neq a \\ 0, & \text{pre } x = a \end{cases} \end{array}\right\} (*)$$

je spojité v bode a pretože

$$\lim_{x \to a} \varepsilon(x) = \lim_{x \to a}\left(\frac{f(x)-f(a)}{x-a} - f'(a)\right) = f'(a) - f'(a) = 0$$

z definície $\varepsilon(x)$ $(*)$ vyplýva, že pre všetky $x \in O_\delta(a)$

$$f(x) - f(a) - f'(a)(x-a) = \varepsilon(x)(x-a)$$

a z toho: $f(x) - f(a) = f'(a)(x-a) + \varepsilon(x)(x-a)$

teda f je diferencovateľná v bode a.

(2) f je diferencovateľná v bode $a \Rightarrow$ existuje $f'(a)$ podľa definície diferencovateľnosti.