Parciálne derivácie funkcie $f \subseteq R^n \times R$

(a) Nech $f \subseteq R^n \times R$ je funkcia
$\bar{a} = (a_1, a_2, \dots, a_n) \in R^n$ a existuje $O_\delta(\bar{a}) \subseteq D(f)$

potom číslo

$$\left[\frac{\partial f(\bar{x})}{\partial x_k}\right]_{\bar{x}=\bar{a}} = \lim_{x_k \to a_k} \frac{f(a_1, a_2, \dots, x_k, a_{k+1}, \dots a_n) - f(a_1, \dots a_n)}{x_k - a_k}$$

ak existuje

sa nazýva **parciálnou deriváciou f v $\bar{a}$ podľa x_k** $(k \in \{1, 2, \dots, n\})$.

$$f(x, y, z) = 3x^2y - 6xz^3 + 15z, \quad \bar{a} = (1, 0, 3)$$

$$\left[\frac{\partial f(x,y,z)}{\partial x}\right]_{(x,y,z)=(1,0,3)} = \left[6xy - 6z^3\right]_{\substack{x=1\\y=0\\z=3}} = -162$$

$$\left[\frac{\partial f(x,y,z)}{\partial y}\right]_{\substack{x=1\\y=0\\z=3}} = \left[3x^2\right]_{\substack{x=1\\y=0\\z=3}} = 3$$

$$\left[\frac{\partial f(x,y,z)}{\partial z}\right]_{\substack{x=1\\y=0\\z=3}} = \left[15 - 18xz^2\right]_{\substack{x=1\\y=0\\z=3}} = -147$$

(pretože ako vyplýva z definície (a) počítame parciálne derivácie podľa jednotlivej premennej x, y alebo z tak že zvyšné 2 premenné pokladáme za konštanty = sú- radnice bodu $\bar{a}$ — a f derivujeme ako funkciu jedinnej premennej).

$$\left[\frac{\partial \sin(x\cdot y)}{\partial x}\right]_{\substack{x=\pi\\y=0}} = \left[\left(\cos(xy)\right)\cdot y\right]_{\substack{x=\pi\\y=0}} = (\cos 0)\cdot 0 = = 0$$

$$\left[\frac{\partial \sin(xy)}{\partial y}\right]_{\substack{x=\pi\\y=0}} = \left[(\cos(xy))\cdot x\right]_{\substack{x=\pi\\y=0}} =$$

$$= \underbrace{\left(\cos(\pi\cdot 0)\right)}_{=1}\cdot \pi = \pi$$

$$f(x,y) = \begin{cases} \dfrac{x^3y - xy^3}{x^2+y^2}, & \text{pre } (x,y) \neq (0,0) \\ 0, & \text{pre } (x,y) = (0,0) \end{cases}$$

$$\left[\frac{\partial f(x,y)}{\partial x}\right]_{\substack{x=0\\y=0}} = \lim_{x\to 0} \frac{f(x,0) - f(0,0)}{x-0} = \lim_{x\to 0} \frac{\frac{0}{x^2} - 0}{x-0} =$$

$$= \lim_{x\to 0} \frac{0}{x} = \lim_{x\to 0} 0 = 0$$

$$\left[\frac{\partial f(x,y)}{\partial y}\right]_{\substack{x=0\\y=0}} = \lim_{y\to 0} \frac{f(0,y) - f(0,0)}{y-0} = \lim_{y\to 0} \frac{\frac{0}{y^2} - 0}{y-0} = 0$$

$$\left[\frac{\partial f(x,y)}{\partial x}\right]_{\substack{x=1\\y=1}} = \left[\frac{(3x^2y - y^3)(x^2+y^2) - (x^3y - xy^3)\cdot 2x}{(x^2+y^2)^2}\right]_{\substack{x=1\\y=1}} = \frac{4}{4} = 1$$

$$\left[\frac{\partial f(x,y)}{\partial y}\right]_{\substack{x=1\\y=1}} = \left[\frac{(x^3 - 3xy^2)(x^2+y^2) - (x^3y - xy^3)\cdot 2y}{(x^2+y^2)^2}\right]_{\substack{x=1\\y=1}} = -1$$

poznámka:

! (v bode (0,0) sa nedal použiť postup ako v bode (1,1), pretože by vychádzal menovateľ rovný nule). Musíme teda použiť definíciu (a).

(b) Funkcia g (označovaná $\frac{\partial f}{\partial x_k}$), $g \subseteq R^n \times R$ sa nazýva **parciálnou deriváciou** (podľa x_k) funkcie $f \subseteq R^n \times R$ na otvorenej množine $G \subseteq D(f)$ ak pre každé $\bar{a} \in G$ existuje $\left[\frac{\partial f}{\partial x_k}\right]_{\bar{x}=\bar{a}} = g(\bar{a})$.

$f(x,y) = \sin(x \cdot y)$ je definovaná na $\mathbb{R}^2$

a má na $\mathbb{R}^2$ $\frac{\partial f}{\partial x} = (\cos xy) \cdot y$ na $\mathbb{R}^2$

$\frac{\partial f}{\partial y} = (\cos xy) \cdot x$ na $\mathbb{R}^2$

a teda napr.: $\left[\frac{\partial [\sin xy]}{\partial x}\right]_{\substack{x=\frac{\pi}{2}\\ y=2}} = \left[(\cos xy) \cdot y\right]_{\substack{x=\frac{\pi}{2}\\ y=2}} =$

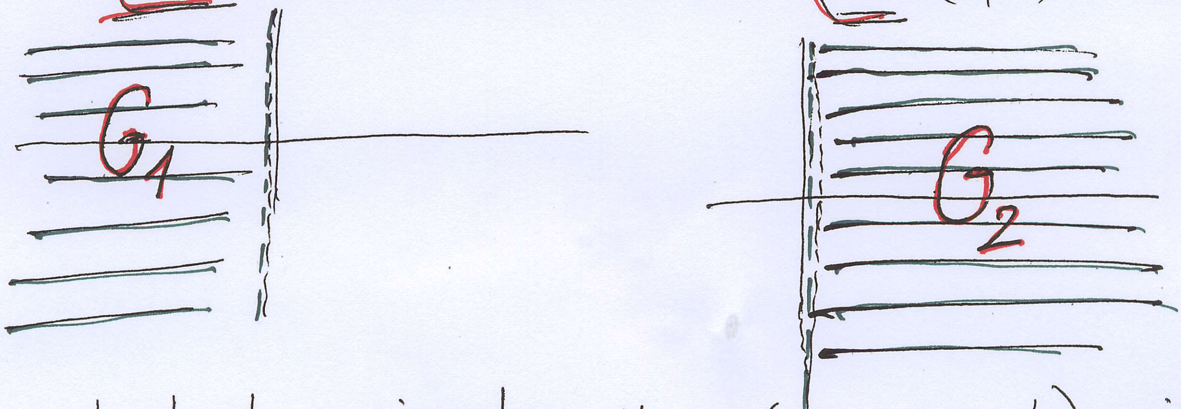
$= \left(\cos\left(\frac{\pi}{2} \cdot 2\right)\right) \cdot 2 = (\cos \pi) \cdot 2 = (-1) \cdot 2 = \underline{-2}$

$\left[\frac{\partial [\sin(xy)]}{\partial y}\right]_{\substack{x=1\\ y=\pi}} = \left[(\cos xy) \cdot x\right]_{\substack{x=1\\ y=\pi}} =$

$= (\cos \pi) \cdot 1 = \underline{-1}$ a podobne v každom bode $(a,b) \in \mathbb{R}^2$

✓ funkcia $f(x,y)=\frac{y}{x}$ je definovaná na otvorených množinách

$G_1=(-\infty,0)\times(-\infty,\infty)$ a $G_2=(0,\infty)\times(-\infty,\infty)$



v bodoch priamky $x=0$ (os y-nová) nie je f definovaná a teda v týchto bodoch nemôže mať parciálnu deriváciu!

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okrem toho $\lim\limits_{\substack{x\to 0\\ y\to 0}} \frac{y}{x}$ neexistuje pretože:

$$\left(\tfrac{1}{k},\tfrac{1}{k}\right)\xrightarrow[k\to\infty]{}(0,0) \text{ ale } f\left(\tfrac{1}{k},\tfrac{1}{k}\right)=\frac{\frac{1}{k}}{\frac{1}{k}}=1\xrightarrow[k\to\infty]{}1$$

$$\left(\tfrac{1}{k},0\right)\xrightarrow[k\to\infty]{}(0,0) \text{ ale } f\left(\tfrac{1}{k},0\right)=\frac{0}{\frac{1}{k}}=0\xrightarrow[k\to\infty]{}0$$

$1\neq 0$

podobne vo všetkých bodoch $(0,a)$ je $\lim\limits_{\substack{x\to 0\\ y\to a}} \frac{y}{x}$ neexistuje, pretože

$$f\Big(\underbrace{\tfrac{1}{k}}_{\to 0},\ \underbrace{a\cdot\tfrac{k+1}{k}}_{\to a}\Big)=\frac{a\cdot\frac{k+1}{k}}{\frac{1}{k}}=a(k+1)\xrightarrow[k\to\infty]{}\infty$$

pre $k\to\infty$

v bodoch priamky $x=0$ sa nedá f dodefinovať na spojitú!

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teda: $\boxed{\frac{\partial f}{\partial x}=\frac{-y}{x^2}}$, $\boxed{\frac{\partial f}{\partial y}=\frac{1}{x}}$ na G_1 a na G_2

(c) Nech funkcia $f \subseteq R^n \times R$ má na nejakom okolí $O_\delta(\bar{a}) \subseteq D(f)$ bodu $\bar{a} \in R^n$ parciálnu deriváciu podľa x_k a je to funkcia $g = \frac{\partial f}{\partial x_k}$. Ak existuje $\frac{\partial g}{\partial x_j}$ v bode $\bar{a}$ potom ju nazývame <u>2-hou parciálnou deriváciou funkcie f v bode $\bar{a}$ podľa x_k a x_j</u> a označujeme $\left[\frac{\partial^2 f}{\partial x_j \partial x_k}\right]_{\bar{x}=\bar{a}}$ $\left(= \left[\frac{\partial g}{\partial x_j}\right]_{\bar{x}=\bar{a}}\right)$.

$$\left(\text{teda } \left[\frac{\partial^2 f}{\partial x_j \partial x_k}\right]_{\bar{x}=\bar{a}} = \left[\frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_k}\right)\right]_{\bar{x}=\bar{a}}\right)$$

2-hou parciálnou deriváciou funkcie f podľa x_k a x_j na otvorenej množine $G \subseteq D(f)$ je funkcia $h \subseteq R^n \times R$ taká, že pre každé $\bar{a} \in G$ je $h(\bar{a}) = \left[\frac{\partial^2 f}{\partial x_j \partial x_k}\right]_{\bar{x}=a}$ (<u>ak existujú</u>)

$f(x,y,z) = 3x^2y - 6xz^3 + 15z$

$\frac{\partial f}{\partial x} = 6xy - 6z^3$ na R^2, $\frac{\partial^2 f}{\partial x^2} = 6y$ na R^2,

$\frac{\partial^2 f}{\partial y \partial x} = 6x$ na R^2, $\frac{\partial^2 f}{\partial z \partial x} = -18z^2$, $\frac{\partial^2 f}{\partial y^2} =$

$= \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(3x^2) = 0$, $\frac{\partial^2 f}{\partial x \partial y} =$

$= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(3x^2) = 6x$, atď...

(d) parciálne derivácie funkcie f vyšších rádov na otvorenej množine $G \subseteq D(f)$ sa definujú podobne ("postupným derivovaním")

- $f(x,y,z) = 3x^2y - 6xz^3 + 15z$

$$\frac{\partial^3 f}{\partial z\,\partial y\,\partial x} = \frac{\partial}{\partial z}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right) = \frac{\partial}{\partial z}\left(\frac{\partial}{\partial y}\underbrace{\left(6xy - 6z^3\right)}_{\frac{\partial f}{\partial x}}\right) =$$

$$= \frac{\partial}{\partial z}(6x) = 0 \text{ na } R^2$$

$$\frac{\partial^3 f}{\partial y\,\partial x^2} = \frac{\partial}{\partial y}\left(\underbrace{\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)}_{\frac{\partial^2 f}{\partial x^2} = 6y}\right) = \frac{\partial}{\partial y}(6y) = 6 \text{ na } R^2$$

$$\frac{\partial^3 f}{\partial x\,\partial y\,\partial x} = \frac{\partial}{\partial x}\left(\underbrace{\frac{\partial^2 f}{\partial y\,\partial x}}_{=6x}\right) = \frac{\partial}{\partial x}(6x) = 6 \text{ na } R^2$$

a podobne. napr.: $\left[\frac{\partial^3 f}{\partial x\,\partial y\,\partial x}\right]_{\substack{x=1\\y=1\\z=1}} = 6$

- $f(x,y) = \frac{y}{x}$ má na množinách $G_1 = (-\infty, 0) \times (-\infty, \infty)$ a $G_2 = (0, \infty) \times (-\infty, \infty)$

parciálne derivácie $\frac{\partial f}{\partial x} = -\frac{y}{x^2}$, $\frac{\partial f}{\partial y} = \frac{1}{x}$

a teda $$\frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial}{\partial y}\left(-\frac{y}{x^2}\right) = -\frac{1}{x^2} \text{ na } G_1 \text{ a na } G_2$$

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$$

na G_1 a na G_2

teda napr.: $\left[\frac{\partial^2 f}{\partial y\,\partial x}\right]_{\substack{x=-3\\y=1}} = \left[-\frac{1}{x^2}\right]_{\substack{x=-3\\y=1}} = -\frac{1}{9}$