Substitúcia $t = tg\frac{x}{2}$

Ak funkcia $f \subseteq R\times R$ vznikne z funkcií $\sin x$, $\cos x$ a konštánt pomocou racionálnych operácií, potom môžeme $\int f(x)dx$ vypočítať pomocou substitúcie $\boxed{t = tg\frac{x}{2}}$

Potom $\sin x = \frac{2t}{t^2+1}$, $\cos x = \frac{1-t^2}{1+t^2}$, $dx = \frac{2}{1+t^2}dt$

$$\int \frac{dx}{\sin x} = \int \frac{\frac{2}{1+t^2}}{\frac{2t}{1+t^2}}dt = \int \frac{1}{t}dt = \ln|\underbrace{tg\frac{x}{2}}_{t}| + C$$

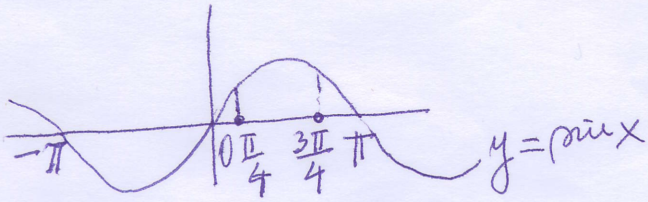
subst: $t = tg\frac{x}{2} \Rightarrow \sin x = \frac{2t}{t^2+1}$, $dx = \frac{2}{1+t^2}dt$

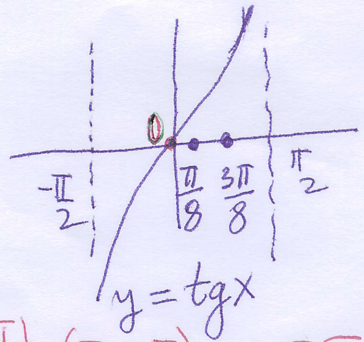
na intervaloch $(k\pi, (k+1)\pi)$, $k = 0, \pm1, \pm2, \dots$

pretože $\sin k\pi = 0$ pre $k = 0, \pm1, \pm2, \dots$ a $tg\frac{k\pi}{2}$ nie je definované pre k-nepárne

napr. $$\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{dx}{\sin x} = \left[\ln\left|tg\frac{x}{2}\right|\right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}} = \ln tg\frac{3\pi}{8} - \ln tg\frac{\pi}{8} = \ln\frac{tg\left(\frac{3\pi}{8}\right)}{tg\left(\frac{\pi}{8}\right)}$$

$\langle \frac{\pi}{4}, \frac{3\pi}{4}\rangle \subseteq (0, \pi)$





ak $x \in \dots, (-2\pi, -\pi), (-\pi, 0), (0, \pi), (\pi, 2\pi), \dots$ teda $\sin x \neq 0$

tak $\frac{x}{2} \in \dots, (-\pi, -\frac{\pi}{2}), (-\frac{\pi}{2}, 0), (0, \frac{\pi}{2}), (\frac{\pi}{2}, \pi), \dots$

$tg\frac{x}{2} \neq 0$

$$\int \frac{1+\cos x}{\sin x}\,dx = \int \frac{1+\frac{1-t^2}{1+t^2}}{\frac{2t}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt =$$

subst. $t = \operatorname{tg}\frac{x}{2}$

$$= \int \frac{1+\cancel{t^2}+1-\cancel{t^2}}{\cancel{2}t} \cdot \frac{\cancel{2}}{1+t^2}\,dt = \int \frac{2}{t(1+t^2)}\,dt = (*)$$

---

rozkladom na parciálne zlomky:

$$\frac{2}{t(1+t^2)} = \frac{A}{t} + \frac{Bt+C}{1+t^2} \quad / \, t(1+t^2)$$

$$2 = A(1+t^2) + t(Bt+C)$$

$t=0$: $\quad 2 = A$

t^2: $\quad 0 = A+B \Rightarrow B=-2$

$t=1$: $\quad 2 = 2A+B+C \Rightarrow 2 = 4-2+C \Rightarrow C=0$

$$\frac{2}{t(1+t^2)} = \frac{2}{t} - \frac{2t}{t^2+1}$$

$$(*) = \int \frac{2}{t}\,dt - \int \frac{2t}{t^2+1}\,dt = 2\ln|t| - \ln(t^2+1) =$$

$$= 2\ln\left|\operatorname{tg}\frac{x}{2}\right| - \ln\left(\operatorname{tg}^2\frac{x}{2}+1\right) + C$$

na intervaloch $(k\pi, (k+1)\pi)$, $k = 0, \pm 1, \pm 2, \dots$

Poznámka:

ak $x \in \dots, (-2\pi, -\pi), (-\pi, 0), (0, \pi), (\pi, 2\pi), \dots$ tak $\sin x \neq 0$

potom $\frac{x}{2} \in \dots, (-\pi, -\frac{\pi}{2}), (-\frac{\pi}{2}, 0), (0, \frac{\pi}{2}), (\frac{\pi}{2}, \pi), \dots$

a teda $\operatorname{tg}\frac{x}{2} \neq 0$ a $\ln\left|\operatorname{tg}\frac{x}{2}\right|$ je definované

Integrovanie racionálnych funkcií
(parciálnych zlomkov)

(1) $a \in R$

$$\int \frac{1}{x-a}\,dx = \ln|x-a| + C$$

na intervaloch $(-\infty, a)$, (a, ∞)

napr.

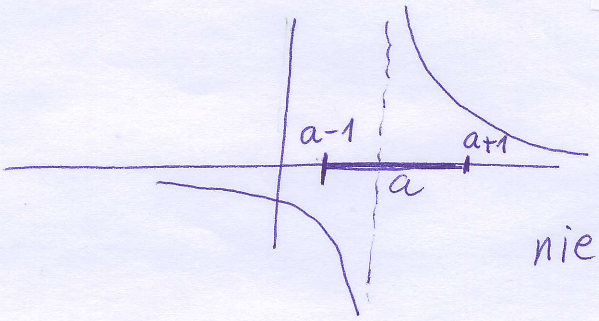
$$\int_{a-3}^{a-1} \frac{1}{x-a}\,dx = \Big[\ln|x-a| + C\Big]_{a-3}^{a-1} =$$

$$= \ln|a-1-a| - \ln|a-3-a| + C =$$

$$= \ln|-1| - \ln|-3| + C = \underline{\ln\frac{1}{3} + C}$$

$$\cancel{\int_{a-1}^{a+1} \frac{1}{x-a}\,dx}$$

nemôžeme uvažovať pretože $\langle a-1, a+1\rangle \not\subseteq (-\infty, a)$, $\not\subseteq (a, \infty)$



$$f(x) = \frac{1}{x-a}$$

nie je na $\langle a-1, a+1\rangle$ ohraničené

✓ $$\int \frac{1}{x-3}\,dx = \ln|x-3| + C \quad \text{na } (-\infty, 3), (3, \infty)$$

✓ $$\int_0^2 \frac{1}{x-3}\,dx = \Big[\ln|x-3|\Big]_0^2 = \underbrace{\ln|-1|}_{0} - \ln|-3| = \underline{\underline{-\ln 3}}$$

pretože $\langle 0, 2\rangle \subseteq (-\infty, 3)$

(2) $a \in R$

$$\int \frac{dx}{(x-a)^n} = \begin{cases} \dfrac{(x-a)^{-n+1}}{-n+1}, \text{ pre } n \neq 1 \\ \text{na } (-\infty, a), (a, \infty) \\ \ln|x-a|, \text{ pre } n = 1 \\ \text{na } (-\infty, a), (a, \infty) \end{cases}$$

• príklad $\displaystyle\int \frac{dx}{(x-3)^4} = \frac{(x-3)^{-4+1}}{-4+1} + C = -\frac{1}{3(x-3)^3} + C$

na int. $(-\infty, 3), (3, \infty)$

(3) $\displaystyle\int \frac{1}{x^2+px+q}\,dx$ / podľa vzorce $\displaystyle\int \frac{dx}{x^2+a^2}$ na $(-\infty, \infty)$

ale $x^2+px+q = (x+\frac{p}{2})^2 + \underbrace{(q-\frac{p^2}{4})}_{>0}$, teda má komplexné korene

• $\displaystyle\int \frac{3}{x^2+2x+5}\,dx = 3\int \frac{dx}{(x+1)^2+4} = 3 \cdot \frac{1}{2} \operatorname{arctg} \frac{x+1}{2} + C$

na intervale $(-\infty, \infty)$

(4) $\displaystyle\int \frac{ax+b}{x^2+px+q}\,dx = \frac{a}{2}\int \frac{2x+\frac{2b}{a}}{x^2+px+q}\,dx =$

$\displaystyle= \frac{a}{2}\int \frac{(2x+p)+(\frac{2b}{a}-p)}{x^2+px+q}\,dx =$

$\displaystyle= \underbrace{\frac{a}{2}\int \frac{2x+p}{x^2+px+q}\,dx}_{=\frac{a}{2}\ln(x^2+px+q)} + \left(\frac{2b}{a}-p\right)\underbrace{\int \frac{dx}{x^2+px+q}}_{\text{ako v (3)}}$

na intervale $(-\infty, \infty)$

$$\int \frac{3x-2}{x^2+4x+6}\,dx = \frac{3}{2}\int \frac{2x-\frac{4}{3}}{x^2+4x+6}\,dx =$$

$$= \frac{3}{2}\int \frac{(2x+4)-\frac{4}{3}-4}{x^2+4x+6}\,dx = \frac{3}{2}\int \frac{2x+4}{x^2+4x+6}\,dx -$$

$$- \frac{16}{3}\cdot\frac{3}{2}\int \frac{dx}{(x+2)^2+2} = \frac{3}{2}\ln\left(x^2+4x+6\right) -$$

$$- 8\cdot\frac{1}{\sqrt{2}}\,\operatorname{arctg}\frac{x+2}{\sqrt{2}} + C \quad \text{na } (-\infty,\infty)$$

napr. $$\int_0^1 \frac{3x-2}{x^2+4x+6}\,dx = \frac{3}{2}\left(\ln 11-\ln 6\right) - \frac{8}{\sqrt{2}}\left(\operatorname{arctg}\frac{3}{\sqrt{2}} - \operatorname{arctg}\frac{2}{\sqrt{2}}\right)$$

$$\int \frac{x+7}{x^2+2x+\cancel{1}}\,dx = \frac{1}{2}\int \frac{2x+14}{x^2+2x+2}\,dx =$$

$$= \frac{1}{2}\int \frac{(2x+2)+12}{x^2+2x+2}\,dx = \frac{1}{2}\int \frac{2x+2}{x^2+2x+2}\,dx +$$

$$+ 6\int \frac{1}{(x+1)^2+1}\,dx = \frac{1}{2}\ln\left(x^2+2x+2\right) +$$

$$+ 6\,\operatorname{arctg}(x+1) + C \quad \text{na } (-\infty,\infty)$$

napr. $$\int_{-1}^{1} \frac{x+7}{x^2+2x+2}\,dx = \frac{1}{2}\left(\ln 5 - \underbrace{\ln 1}_{0}\right) +$$

$$+ 6\left(\operatorname{arctg}2 - \underbrace{\operatorname{arctg}0}_{0}\right) = \frac{1}{2}\ln 5 + 6\operatorname{arctg}2$$

(5) $\int \frac{1}{(x^2+1)^2}\,dx$ môžeme vypočítať použitím "per partes" na integrál

$$\int \frac{1}{x^2+1}\,dx = \int \underbrace{1}_{f'} \cdot \underbrace{\frac{1}{x^2+1}}_{g}\,dx = \underbrace{x}_{f} \cdot \underbrace{\frac{1}{x^2+1}}_{g} - \int \underbrace{x}_{f} \underbrace{(-1)\frac{2x}{(x^2+1)^2}}_{g'}\,dx =$$

$$= \frac{x}{x^2+1} + 2\int \frac{x^2}{(x^2+1)^2}\,dx = \frac{x}{x^2+1} + 2\int \frac{(x^2+1)-1}{(x^2+1)^2}\,dx$$

$$= \frac{x}{x^2+1} + 2\int \frac{1}{x^2+1}\,dx - 2\int \frac{1}{(x^2+1)^2}\,dx$$

z podtrhnutej rovnice je

$$2\int \frac{1}{(x^2+1)^2}\,dx = \frac{x}{x^2+1} + \int \frac{1}{x^2+1}\,dx$$

$$\boxed{\int \frac{1}{(x^2+1)^2}\,dx = \frac{1}{2}\frac{x}{x^2+1} + \frac{1}{2}\operatorname{arctg} x + C}$$

na $(-\infty, \infty)$

Rovnakou metódou môžeme vypočítať

$\int \frac{1}{(x^2+1)^3}\,dx$ použitím "per partes" na integrál

$$\int \underbrace{1}_{f'} \cdot \underbrace{\frac{1}{(x^2+1)^2}}_{g}\,dx = \cdots$$

podobne pre $n = 4, 5, 6, \ldots$

Do $\int \frac{1}{(x^2+a^2)^n}\,dx$ ak $n > 1$ zavedieme najprv substitúciu $x = at \Rightarrow dx = a\,dt$

$$= \int \frac{1}{(a^2t^2+a^2)^n}\,a\,dt = \frac{a}{a^{2n}}\int \frac{1}{(t^2+1)^n}\,dt \Big|_{t=\frac{x}{a}} = \cdots$$

(6) $$\int \frac{ax+b}{(x^2+1)^n}\,dx = \frac{a}{2}\int \frac{2x+\frac{2b}{a}}{(x^2+1)^n}\,dx =$$

$$= \underbrace{\frac{a}{2}\int \frac{2x}{(x^2+1)^n}\,dx}_{\text{substitúcia}} + b\int \frac{1}{(x^2+1)^n}\,dx =$$

substitúcia
$t = x^2+1 \Rightarrow 2x\,dx = dt$

$$= \frac{a}{2}\int \frac{dt}{t^n}\Big|_{t=x^2+1} + b\int \frac{1}{(x^2+1)^n}\,dx =$$

$$= \frac{a}{2}\,\frac{t^{-n+1}}{-n+1} + b\int \frac{1}{(x^2+1)^n}\,dx =$$

$$= \frac{a}{2}\,\frac{1}{(1-n)(x^2+1)^{n-1}} + \underbrace{b\int \frac{1}{(x^2+1)^n}\,dx}_{\text{použitím (5)}}$$

$$\int \frac{2x-1}{(x^2+2x+2)^2}\,dx = \int \frac{2x+2}{(x^2+2x+2)^2}\,dx - 3\underbrace{\int \frac{1}{((x+1)^2+1)^2}\,dx}_{\text{použitím (5)}} =$$

$$= -\frac{1}{x^2+2x+2} - \frac{3}{2}\,\frac{x+1}{x^2+2x+2} - \frac{3}{2}\,\mathrm{arctg}\,(x+1) + C$$

na $(-\infty,\infty)$

napr. vypočítajte

$$\int_0^1 \frac{2x-1}{(x^2+2x+2)^2}\,dx$$ použitím predchádzajúceho výsledku.

$$\int \frac{1}{x^4+1}\,dx = \qquad \text{na intervalu } (-\infty, \infty)$$

$$x^4+1 = (x^2+1)^2 - 2x^2 = \left((x^2+1)-\sqrt{2}x\right)\left((x^2+1)+\sqrt{2}x\right)$$

$$\frac{1}{x^4+1} = \frac{ax+b}{x^2+\sqrt{2}x+1} + \frac{cx+d}{x^2-\sqrt{2}x+1} \Big/ (x^4+1)$$

$$1 = (ax+b)(x^2-\sqrt{2}x+1) + (cx+d)(x^2+\sqrt{2}x+1)$$

x^3: $0 = a + c \Rightarrow c = -a$

$x=0$: $1 = b + d \Rightarrow d = (1-b)$

x^1: $0 = a + c - \sqrt{2}b + \sqrt{2}d$

$0 = -\sqrt{2}b + \sqrt{2}(1-b) \Rightarrow \sqrt{2} - 2\sqrt{2}b = 0 \Big/ \frac{1}{\sqrt{2}}$

$1 - 2b = 0 \Rightarrow b = \frac{1}{2}$

$d = 1 - b = \frac{1}{2} \Rightarrow d = \frac{1}{2}$

x^2: $0 = \underbrace{b+d} - \sqrt{2}a + \sqrt{2}c$

$0 = 1 - \sqrt{2}a - \sqrt{2}a \Rightarrow 1 = 2\sqrt{2}a \Rightarrow a = \frac{1}{2\sqrt{2}}$

a teda $c = -\frac{1}{2\sqrt{2}}$

$$\int \frac{1}{x^4+1}\,dx = \frac{1}{2\sqrt{2}}\int \frac{x + \frac{1}{2}2\sqrt{2}}{x^2+\sqrt{2}x+1}\,dx - \frac{1}{2\sqrt{2}}\int \frac{x - \frac{1}{2}2\sqrt{2}}{x^2-\sqrt{2}x+1}\,dx =$$

$$= \frac{1}{4\sqrt{2}}\int \frac{(2x+\sqrt{2})+\sqrt{2}}{x^2+\sqrt{2}x+1}\,dx - \frac{1}{4\sqrt{2}}\int \frac{(2x-\sqrt{2})-\sqrt{2}}{x^2-\sqrt{2}x+1}\,dx =$$

$$= \frac{1}{4\sqrt{2}}\ln(x^2+\sqrt{2}x+1) + \frac{1}{4}\int \frac{dx}{\left(x+\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}} - \frac{1}{4\sqrt{2}}\ln(x^2-\sqrt{2}x+1) +$$

$$+ \frac{1}{4}\int \frac{dx}{\left(x-\frac{\sqrt{2}}{2}\right)^2+\frac{1}{2}} = \frac{1}{4\sqrt{2}}\ln\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1} +$$

$$+ \frac{1}{4}\cdot\sqrt{2}\,\mathrm{arctg}\,\frac{x+\frac{\sqrt{2}}{2}}{\frac{1}{\sqrt{2}}} + \frac{1}{4}\sqrt{2}\,\mathrm{arctg}\,\frac{x-\frac{\sqrt{2}}{2}}{\frac{1}{\sqrt{2}}} + C$$

$\frac{x+\frac{\sqrt{2}}{2}}{\frac{1}{\sqrt{2}}} = \sqrt{2}x+1$, $\quad \frac{x-\frac{\sqrt{2}}{2}}{\frac{1}{\sqrt{2}}} = \sqrt{2}x-1$

$$\int \frac{1}{x^6+1}\,dx$$

rovnica $x^6+1=0$ má korene $i, -i$ a teda môžeme deliť x^6+1 súčinom $(x-i)(x+i)=x^2+1$ dostaneme

$$x^6+1=(x^2+1)(x^4-x^2+1)=$$
$$=(x^2+1)((x^2+1)^2-3x^2)=$$
$$=(x^2+1)(x^2+1-\sqrt{3}\,x)(x^2+1+\sqrt{3}\,x)$$ má len komplex. korene

a rozkladom na parciálne zlomky

(*) $$\frac{1}{x^6+1}=\frac{ax+b}{x^2+1}+\frac{cx+d}{x^2+\sqrt{3}x+1}+\frac{ex+f}{x^2-\sqrt{3}x+1}$$

môžeme vypočítať daný integrál. na intervale $(-\infty,\infty)$

najdite a, b, c, d, e, f a vypočítajte daný integrál.

---

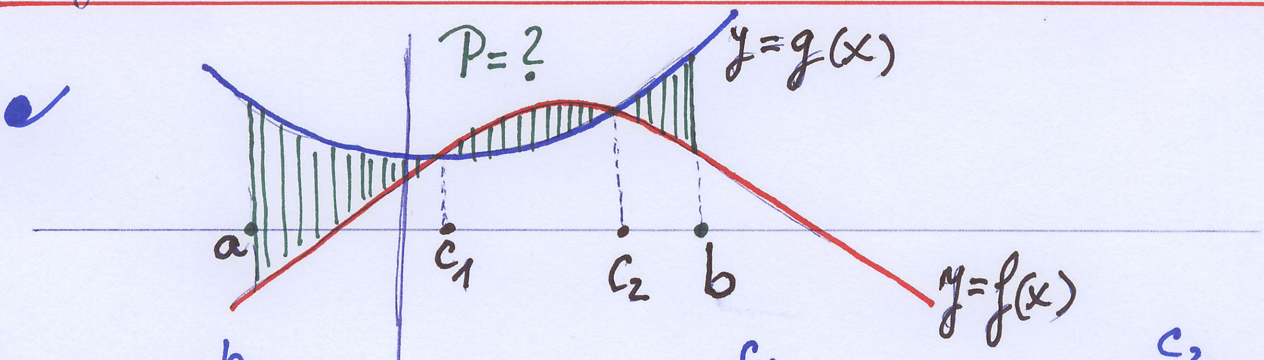
urobte skúšku rozkladu! (*)

urobte skúšku dosiahnutého výsledku!

Výpočet plošných obsahov

Ak $f, g \subseteq R \times R$ sú spojité funkcie na $\langle a, b\rangle$

potom $\boxed{\int_a^b |f(x)-g(x)|\,dx}$ nazývame <u>plošným obsahom</u> rovinnej oblasti ohraničenej grafmi funkcií f, g a priamkami $x=a$, $x=b$.



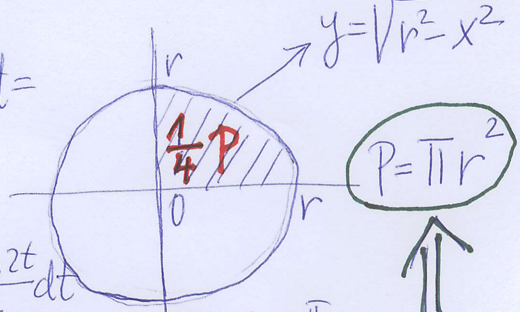
$$P=\int_a^b |f(x)-g(x)|\,dx = \int_a^{c_1}(g(x)-f(x))\,dx + \int_{c_1}^{c_2}(f(x)-g(x))\,dx +$$
$$+\int_{c_2}^{b}(g(x)-f(x))\,dx$$

Pr. Vypočítajte plochu kružnice o polomere $r>0$.

$$\boxed{\frac{1}{4}P} = \int_0^r \sqrt{r^2-x^2}\,dx = \int_0^{\frac{\pi}{2}} \sqrt{r^2-r^2\sin^2 t}\; r\cos t\,dt =$$
$$= r^2\int_0^{\frac{\pi}{2}} \cos^2 t\,dt$$
$$= r^2\int_0^{\frac{\pi}{2}} \frac{1+\cos 2t}{2}\,dt$$
$$= r^2\left[\frac{1}{2}t + \frac{\sin 2t}{4}\right]_0^{\frac{\pi}{2}} =$$
$$= r^2\left(\frac{\pi}{4}+0-0\right) = \boxed{\frac{\pi}{4}r^2}$$

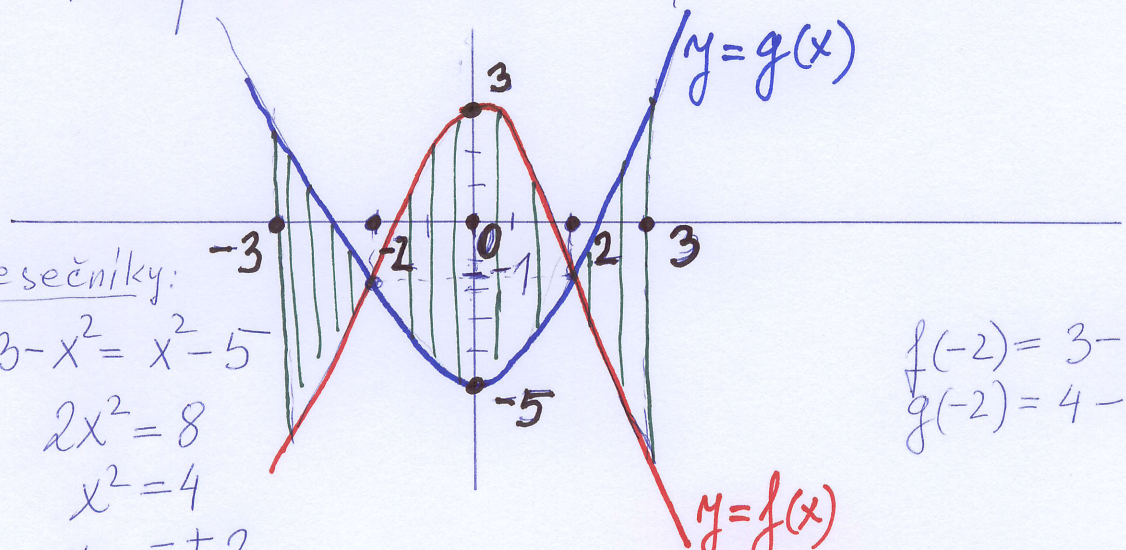
subst: $x = r\sin t$
$dx = r\cos t\,dt$

$x=0 \Rightarrow 0 = r\sin t \Rightarrow \sin t = 0 \Rightarrow t=0$
$x=r \Rightarrow r = r\sin t \Rightarrow \sin t = 1 \Rightarrow t=\frac{\pi}{2}$



$$P=\pi r^2$$

Vypočítajte plošný obsah ohraničený grafmi funkcií $f(x)=3-x^2$, $g(x)=x^2-5$ a priamkami $x=3$, $x=-3$.



priesečníky:

$3-x^2=x^2-5$

$2x^2=8$

$x^2=4$

$x_{1,2}=\pm 2$

$f(-2)=3-4=-1=f(2)$

$g(-2)=4-5=-1=g(2)$

$$P=2\left\{\int_0^2 (f(x)-g(x))dx+\int_2^3 (g(x)-f(x))dx\right\}=$$

$$=2\left\{\int_0^2 ((3-x^2)-(x^2-5))dx+\int_2^3 ((x^2-5)-(3-x^2))dx\right\}=$$

$$=2\left\{\int_0^2 (8-2x^2)dx+\int_2^3 (2x^2-8)dx\right\}=$$

$$=2\left\{\left[8x-2\frac{x^3}{3}\right]_0^2+\left[2\frac{x^3}{3}-8x\right]_2^3\right\}=$$

$$=\cancel{2\left\{16-\frac{16}{3}+2\frac{27}{3}\right.}$$

$$=2\left\{\left(16-\frac{16}{3}-0\right)+\left(2\frac{27}{3}-24-\left(2\frac{8}{3}-16\right)\right)\right\}=$$

$$=2\left\{\frac{32}{3}+18-24-\frac{16}{3}+16\right\}=2\left\{\frac{16}{3}+10\right\}=$$

$$=\boxed{\frac{92}{3}}$$