Substitučná metóda

Zopakujme si:

(1) Substitúcia $t=\varphi(x)$ do $\int f(\varphi(x))\cdot\varphi'(x)dx$:

Nech
- F je primitívna k f na (α,β)
- φ má deriváciu φ' na (a,b)
- pre každé $x\in(a,b)$ je $\varphi(x)\in(\alpha,\beta)$

potom $\int f(\varphi(x))\cdot\varphi'(x)\,dx = \int f(t)\,dt\Big|_{t=\varphi(x)} = F(\varphi(x))+C$ na (a,b)

pretože:
$([F(\varphi(x))]' = f(\varphi(x))\cdot\varphi'(x)$ pre každé $x\in(a,b))$

$$\int\frac{1}{x\ln x}dx = \int\underbrace{\frac{1}{\ln x}}_{f(\varphi(x))}\cdot\underbrace{\frac{1}{x}}_{\varphi'(x)}dx = \int\underbrace{\frac{1}{t}}_{f(t)}dt = \ln|t|\Big|_{t=\ln x} = \ln|\ln x|+C$$

$t=\ln x$
$dt=\frac{1}{x}dx$

na intervaloch $(0,1)$, $(1,\infty)$

napr.
$$\int_{\frac12}^{\frac34}\frac{1\,dx}{x\ln x} = \Big[\ln|\ln x|\Big]_{\frac12}^{\frac34} = \ln\left(\left|\ln\tfrac34\right|\right)-\ln\left(\left|\ln\tfrac12\right|\right) = \ln\left(\ln\tfrac43\right)-\ln(\ln 2)$$

(pretože: $\left|\ln\frac34\right| = |\ln 3-\ln 4| = \ln 4-\ln 3 = \ln\frac43$

a podobne $\left|\ln\frac12\right| = |\underbrace{\ln 1}_{0}-\underbrace{\ln 2}_{>0}| = \ln 2$) pretože $\langle\frac12,\frac34\rangle\subseteq(0,1)$

napr.
$$\int_2^e\frac{1}{x\ln x}dx = \Big[\ln|\ln x|\Big]_2^e = \ln|\underbrace{\ln e}_{1}|-\ln|\ln 2| = \underbrace{\ln 1}_{0}-\ln(\ln 2) = -\ln(\ln 2)$$

pretože $\langle 2,e\rangle\subseteq(1,\infty)$

$\cancel{\int_{\frac12}^{\frac32}\frac{1}{x\ln x}dx}$ nemôžeme uvažovať pretože $\langle\frac12,\frac32\rangle\not\subseteq(0,1)$ ani $(1,\infty)$

$$\int \frac{2x}{\sqrt{3x^2+9}}\,dx = \frac{1}{3}\int \frac{6x}{\sqrt{3x^2+9}}\,dx = \frac{1}{3}\int \frac{2t}{\sqrt{t^2}}\,dt = \frac{1}{3}\int 2\,dt =$$

$t^2 = 3x^2+9 \Rightarrow t = \sqrt{3x^2+9}$

$2t\,dt = 6x\,dx$

$$= \frac{2}{3}t + C = \frac{2}{3}\sqrt{3x^2+9} + C$$

na intervale $(-\infty,\infty)$

(skúška: $\left[\frac{2}{3}\sqrt{3x^2+9}\right]' = \frac{2}{3}\cdot\frac{1}{2\sqrt{3x^2+9}}\cdot 6x = \frac{2x}{\sqrt{3x^2+9}}$

na intervale $(-\infty,\infty)$)

$$\int (3x+1)\sqrt{3x-1}\,dx = \int (t+2)\sqrt{t}\cdot\frac{1}{3}\,dt = \frac{1}{3}\int \left(t^{\frac{3}{2}} + 2t^{\frac{1}{2}}\right)dt =$$

$\left(\begin{array}{l} t = 3x-1 \Rightarrow 3x+1 = t+2 \\ dt = 3\,dx \Rightarrow \frac{1}{3}dt = dx \end{array}\right)$

$$= \frac{1}{3}\left(\frac{2t^{\frac{5}{2}}}{5} + 2\frac{2t^{\frac{3}{2}}}{3}\right) + C = \frac{1}{3}\left(\frac{2}{5}\sqrt{(3x-1)^5} + \frac{4}{3}\sqrt{(3x-1)^3}\right) + C$$

na intervale $\left(\frac{1}{3},\infty\right)$

napr. $$\int_1^3 (3x+1)\sqrt{3x-1}\,dx = \left[\frac{1}{3}\left(\frac{2}{5}\sqrt{(3x-1)^5} + \frac{4}{3}\sqrt{(3x-1)^3}\right)\right]_1^3 =$$

$$= \frac{1}{3}\left(\frac{2}{5}\cdot 8^{\frac{5}{2}} + \frac{4}{3}\cdot 8^{\frac{3}{2}}\right),$$ pretože $\langle 1,3\rangle \subseteq \left(\frac{1}{3},\infty\right)$

napr. $$\cancel{\int_{-1}^{0} (3x+1)\sqrt{3x-1}\,dx}$$ nemôžeme uvažovať pretože $\langle -1,0\rangle \not\subseteq \left(\frac{1}{3},\infty\right)$

$\sqrt{3x-1}$ je definované pre $3x-1>0$

$x > \frac{1}{3}$

$\int tg\,x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{(-\sin x)}{\cos x}\,dx = -\ln|\cos x| + C$

subst.: $t=\cos x$ na intervaloch $\left(-\frac{\pi}{2}+k\cdot\pi, \frac{\pi}{2}+k\cdot\pi\right)$, $k=0,\pm1,\pm2,\dots$

teda $\dots\left(-\frac{5\pi}{2},-\frac{3\pi}{2}\right), \left(-\frac{3\pi}{2},-\frac{\pi}{2}\right), \left(-\frac{\pi}{2},\frac{\pi}{2}\right), \left(\frac{\pi}{2},\frac{3\pi}{2}\right), \left(\frac{3\pi}{2},\frac{5\pi}{2}\right),\dots$

napr.: $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} tg\,x\,dx = \left[\ln|\cos x|\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \ln\left(\cos\frac{\pi}{4}\right) - \ln\left(\cos\left(-\frac{\pi}{4}\right)\right) =$

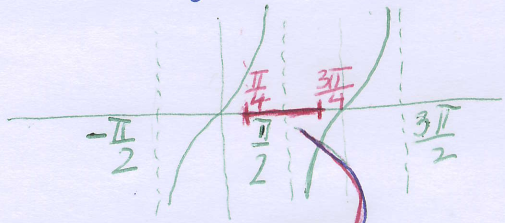
$= \ln\frac{\sqrt{2}}{2} - \ln\frac{\sqrt{2}}{2} = 0$

$\int_{0}^{\frac{\pi}{4}} tg\,x\,dx = \left[\ln \cos x\right]_{0}^{\frac{\pi}{4}} = \ln\left(\cos\frac{\pi}{4}\right) - \ln(\cos 0) =$

$= \ln\frac{\sqrt{2}}{2} - \underbrace{\ln 1}_{0} = \ln\frac{\sqrt{2}}{2}$

pretože $\left\langle -\frac{\pi}{4}, \frac{\pi}{4}\right\rangle \subseteq \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\left\langle 0, \frac{\pi}{4}\right\rangle \subseteq \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



určitý R-int.: $\cancel{\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} tg\,x\,dx}$ nemôžeme uvažovať

pretože $\left\langle \frac{\pi}{4}, \frac{3\pi}{4}\right\rangle \not\subseteq \left(-\frac{\pi}{2}+k\pi, \frac{\pi}{2}+k\pi\right)$

pre žiadne celé k

$f(x)=tg\,x$ je tam neohraničená!

$\int cotg\,x\,dx = \int \frac{\cos x}{\sin x}\,dx = \cdots$ riešime podobne

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$\int \sin^4 x \cos x\,dx = \int t^4\,dt = \frac{t^5}{5}\Big|_{t=\sin x} = \frac{\sin^5 x}{5} + C$

subst.: $t = \sin x$

$dt = \cos x\,dx$

na intervale $(-\infty, \infty)$

$\int \sin^3 x \cos^4 x\,dx = \int \sin^2 x \cos^4 x \sin x\,dx =$

subst.: $t = \cos x$

$dt = -\sin x\,dx$

$(-1)dt = \sin x\,dx$

$= \int (1-\cos^2 x)\cos^4 x \sin x\,dx =$

$= \int (1-t^2)\,t^4\,(-1)\,dt =$

$= \int (-t^4+t^6)\,dt = -\frac{t^5}{5} + \frac{t^7}{7} + C =$

$= -\frac{1}{5}\cos^5 x + \frac{1}{7}\cos^7 x + C$ na $(-\infty, \infty)$

substitúcia $x = \varphi(t)$ do $\int f(x)dx$

zopakujme si:

Nech
- f je spojitá na (a,b)
- pre všetky $t \in (\alpha,\beta)$ je $\varphi(t) \in (a,b)$
  a existuje $\varphi'(t) \neq 0$ pre všetky $t \in (\alpha,\beta)$
  spojitá φ' na (α,β)
- na (a,b) nech $t = \varphi^{-1}(x)$ je inverzná k φ

potom
$$\int f(x)dx = \int f(\varphi(t))\varphi'(t)dt \Big|_{t=\varphi^{-1}(x)}$$

- $\underline{a>0}$ $$\int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{a\,dt}{\sqrt{a^2-a^2t^2}} = \int \frac{a}{a\sqrt{1-t^2}}dt = \int \frac{1}{\sqrt{1-t^2}}dt =$$

substit.: $x = at \Rightarrow t = \frac{x}{a}$
$dx = a\,dt$

$$= \arcsin t \Big|_{t=\frac{x}{a}} + C = \arcsin\frac{x}{a} + C$$

na $(-a,a)$

- $$\int 3\sqrt{1-x^2}\,dx = \int 3\sqrt{1-\sin^2 t}\cos t\,dt = 3\int \cos^2 t\,dt$$

subst.: $x = \sin t$
$dx = \cos t\,dt$
$\Rightarrow t = \arcsin x$
na $(-1,1)$

$$= 3\int \frac{1+\cos 2t}{2}dt = \frac{3}{2}\int (1+\cos 2t)dt =$$
$$= \frac{3}{2}\left(t + \frac{\sin 2t}{2}\right)\Big|_{t=\arcsin x} =$$
$$= \frac{3}{2}\left(\arcsin x + \frac{1}{\cancel{2}}\cancel{2}x\sqrt{1-x^2}\right)$$
$$= \frac{3}{2}\left(\arcsin x + x\sqrt{1-x^2}\right)$$

na intervale $(-1,1)$

pretože $\sin 2t = 2\sin t\cos t \Big|_{t=\arcsin x}$
$= 2\sin(\arcsin x)\sqrt{1-\sin^2(\arcsin x)}$
$= 2x\sqrt{1-x^2}$

Substitučná metóda pre určité integrály

Zopakujme si: substitúcia $t=\varphi(x)$

Nech
- f je spojitá na $\langle \alpha, \beta \rangle$
- φ má spojitú deriváciu φ' na $\langle a, b \rangle$
- pre každé $x \in \langle a, b \rangle$ je $\varphi(x) \in \langle \alpha, \beta \rangle$

pričom $\varphi(a) = \alpha$, $\varphi(b) = \beta$

potom $$\int_a^b f(\varphi(x))\,\varphi'(x)\,dx = \int_{\alpha=\varphi(a)}^{\beta=\varphi(b)} f(t)\,dt$$

$$\int_1^e \frac{\ln^3 x}{x}\,dx = \int_1^e (\ln^3 x)\cdot\frac{1}{x}\,dx = \int_{0=\ln 1}^{1=\ln e} t^3\,dt = \left[\frac{t^4}{4}\right]_0^1 = \frac{1}{4}$$

subst: $t = \ln x \Rightarrow \ln^3 x = t^3$

$dt = \frac{1}{x}\,dx$

ak $x=1$ tak $t=\ln 1 = 0$

ak $x=e$ tak $t = \ln e = 1$

$x \in \langle 1, e \rangle \Leftrightarrow \ln x \in \langle 0, 1 \rangle$

$$\int_0^1 \frac{e^{2x}}{\sqrt[4]{e^x+1}}\,dx = \int_0^1 \frac{e^x}{\sqrt[4]{e^x+1}}\,e^x\,dx = \int_{\sqrt[4]{2}=\sqrt[4]{e^0+1}}^{\sqrt[4]{e+1}=\sqrt[4]{e^1+1}} \frac{t^4-1}{t}\,4t^3\,dt =$$

$t = \sqrt[4]{e^x+1}$

$t^4 = e^x + 1 \Rightarrow e^x = t^4 - 1$

$4t^3\,dt = e^x\,dx$

$$= \int_{\sqrt[4]{2}}^{\sqrt[4]{e+1}} \left(t^3 - \frac{1}{t}\right) 4t^3\,dt$$

$$= 4\left[\frac{1}{7}t^7 - \frac{1}{3}t^3\right]_{\sqrt[4]{2}}^{\sqrt[4]{e+1}} = \text{atd}$$

$$\int \sqrt{5-4x-x^2}\,dx = \int \sqrt{9-(x+2)^2}\,dx =$$

<u>subst:</u> $x+2 = 3\sin t \Rightarrow t = \arcsin\frac{x+2}{3}$

$dx = 3\cos t\,dt$

$$= \int \underbrace{\sqrt{9-9\sin^2 t}}_{3\cos t} \cdot 3\cos t\,dt = \int 9\cdot\cos^2 t\,dt =$$

$$= 9\int \frac{1+\cos 2t}{2}\,dt = \frac{9}{2}\left(t+\frac{\sin 2t}{2}\right)\Big/_{t=\arcsin\frac{x+2}{3}} =$$

$$\frac{9}{2}\left(\arcsin\frac{x+2}{3} + \frac{x+2}{3}\sqrt{1-\left(\frac{x+2}{3}\right)^2}\right) + C$$

$$\frac{\sin 2t}{2} = \frac{2\sin t\cos t}{2} = \sin t\cos t =$$

$$= (\sin t)\sqrt{1-\sin^2 t}$$

na intervale:

$(-5, 1)$

$$9-(x+2)^2 > 0$$
$$(x+2)^2 < 9$$
$$|x+2| < 3$$
$$-3 < x+2 < 3$$
$$-5 < x < 1$$

$$\int_{-\pi}^{0} \sqrt{5-4x-x^2}\,dx = \int_{-\pi}^{0} \sqrt{9-(x+2)^2}\,dx = \int_{\arcsin\frac{-\pi+2}{3}}^{\arcsin\frac{2}{3}} 9\cos^2 t\,dt =$$

<u>subst.:</u> $x+2 = 3\sin t$

$dx = 3\cos t$

$$= \left[\frac{9}{2}\left(t+\frac{\sin 2t}{2}\right)\right]_{\arcsin\frac{-\pi+2}{3}}^{\arcsin\frac{2}{3}}$$

$$= \frac{9}{2}\left(\arcsin\frac{2}{3} - \arcsin\frac{-\pi+2}{3} + \frac{2}{3}\sqrt{1-\left(\frac{2}{3}\right)^2} - \left(\frac{-\pi+2}{3}\right)\cdot\sqrt{1-\left(\frac{-\pi+2}{3}\right)^2}\right)$$

$$\int \frac{1}{2\sqrt{x}} \cos\sqrt{x}\, dx = \int \cos t\, dt = \sin t + C \Big/_{t=\sqrt{x}} =$$

subst.: $t = \sqrt{x}$

$dt = \frac{1}{2\sqrt{x}} dx$

$$= \underline{\sin\sqrt{x} + C}$$

na intervale $(0, \infty)$

$$\int_{\pi^2}^{\pi^4} \frac{1}{2\sqrt{x}} \cos\sqrt{x}\, dx = \int_{\pi=\sqrt{\pi^2}}^{\pi^2=\sqrt{\pi^4}} \cos t\, dt = \Big[\sin t\Big]_{\pi}^{\pi^2} =$$

subst.: $t = \sqrt{x}$

$$= \underline{\sin\pi^2 - \sin\pi}$$

$$\int \cos\sqrt{x}\, dx = \int \underbrace{(\cos t)}_{f'} . \underbrace{2t}_{g'}\, dt = \underbrace{(\sin t)}_{f} . \underbrace{2t}_{g} - \int \underbrace{(\sin t)}_{f} \underbrace{2}_{g'}\, dt =$$

subst.: $x = t^2$

$dx = 2t\,dt$

$$= 2t \sin t + 2\cos t \Big/_{t=\sqrt{x}} = \underline{2\sqrt{x}\sin\sqrt{x} +}$$

$$\underline{+ 2\cos\sqrt{x} + C}$$

na intervale $(0, \infty)$

$$\int \frac{x}{(x^2+1)^2} dx = \frac{1}{2}\int \frac{2x\,dx}{(x^2+1)^2} = \frac{1}{2}\int \frac{dt}{t^2} = \frac{1}{2}\frac{t^{-1}}{-1} =$$

subst.: $x^2+1 = t$

$2x\,dx = dt$

$$= -\frac{1}{2t}\Big/_{t=x^2+1} = -\frac{1}{2(x^2+1)} + C$$

na intervale $(-\infty, \infty)$

$$\int_0^1 \frac{x}{(x^2+1)^2} dx = \frac{1}{2}\int_1^2 \frac{dt}{t^2} = \Big[-\frac{1}{2t}\Big]_1^2 = -\frac{1}{4} + \frac{1}{2} = \underline{\underline{\frac{1}{4}}}$$

subst.: $x^2+1 = t$

$2x\,dx = dt$