

n-rozmerne integrály - R-integrál z ohraničenej funkcie $f \subseteq \mathbb{R}^n \times \mathbb{R}$ na n-kvádri $E(\bar{a}, \bar{b})$

$(E(\bar{a}, \bar{b}) = \langle a_1, b_1 \rangle \times \langle a_2, b_2 \rangle \times \dots \times \langle a_n, b_n \rangle)$
je uzavretý ohraničený interval v \mathbb{R}^n .

$\int_{E(\bar{a}, \bar{b})} l \, d\bar{x} = l \cdot C(E(\bar{a}, \bar{b}))$, $l \in \mathbb{R}$ dané
 $= l(b_1 - a_1)(b_2 - a_2) \dots (b_n - a_n)$

pretože, z definície Riemannovho integrálu je $\int_{E(\bar{a}, \bar{b})} k \, d\bar{x} = J$, jediné číslo medzi všetkými dolnými

integrálnymi súčtami $D(f; \mathcal{D}, E(\bar{a}, \bar{b})) = \sum_{k=1}^m m_k C(E_k)$

a všetkými hornými integrálnymi súčtami $H(f; \mathcal{D}, E(\bar{a}, \bar{b})) = \sum_{k=1}^m M_k C(E_k)$, (ak existuje J jediné!)

\mathcal{D} je delenie $E(\bar{a}, \bar{b})$ na čiastočné n-kv. E_1, E_2, \dots, E_m
 $E = E(\bar{a}, \bar{b}) = \bigcup_{k=1}^m E_k$, $C(E) = (b_1 - a_1)(b_2 - a_2) \dots (b_n - a_n)$
 $= \sum_{k=1}^m C(E_k)$, n-rozmerý obsah E.

$m_k = \inf \{ f(\bar{x}) \mid \bar{x} \in E_k \}$ $M_k = \sup \{ f(\bar{x}) \mid \bar{x} \in E_k \}$ $f \subseteq \mathbb{R}^n \times \mathbb{R}$ daná funkcia n-premených

ak $f(\bar{x}) = l$ pre všetky $\bar{x} \in E(\bar{a}, \bar{b}) = \bigcup_{k=1}^m E_k$, $l \in \mathbb{R}$
potom $m_k = M_k = l$ a teda pre každé \mathcal{D}

je $D(f; \mathcal{D}, E(\bar{a}, \bar{b})) = \sum_{k=1}^m l C(E_k) = l \sum_{k=1}^m C(E_k) = l \cdot C(E(\bar{a}, \bar{b}))$
 $H(f; \mathcal{D}, E(\bar{a}, \bar{b})) = \sum_{k=1}^m l C(E_k) = l \sum_{k=1}^m C(E_k) = l \cdot C(E(\bar{a}, \bar{b}))$
 $\Rightarrow \int_{E(\bar{a}, \bar{b})} l \, d\bar{x} = l \cdot C(E(\bar{a}, \bar{b}))$ (existuje jediné!)

$$\iiint (x^2z + y) dx dy dz = \iiint x^2z dx dy dz + \iiint y dx dy dz$$

$\langle 0,1 \rangle \times \langle 1,2 \rangle \times \langle -1,2 \rangle$ $\langle 0,1 \rangle \times \langle 1,2 \rangle \times \langle -1,2 \rangle$

$$+ \iiint y dx dy dz = \left(\int_0^1 x^2 dx \right) \left(\int_1^2 1 dy \right) \left(\int_{-1}^2 z dz \right) + \left(\int_0^1 1 dx \right) \left(\int_1^2 y dy \right) \left(\int_{-1}^2 1 dz \right)$$

$\langle 0,1 \rangle \times \langle 1,2 \rangle \times \langle -1,2 \rangle$ $\langle 0,1 \rangle \times \langle 1,2 \rangle \times \langle -1,2 \rangle$

$$= \frac{1}{2} + [x]_0^1 \left[\frac{y^2}{2} \right]_1^2 \left[\frac{z^2}{2} \right]_{-1}^2 = \frac{1}{2} + \frac{9}{2} = \frac{10}{2}$$

pretože; podľa vety o výpočte

je: $\iiint x^2z dx dy dz = \int_1^2 \left(\iint x^2z dx dz \right) dy =$

$\langle 0,1 \rangle \times \langle 1,2 \rangle \times \langle -1,2 \rangle$ $\langle 0,1 \rangle \times \langle 1,2 \rangle$

$$= \int_1^2 \left(\int_0^1 \left(\int_{-1}^2 x^2z dz \right) dx \right) dy = \int_1^2 \left(\int_0^1 x^2 \left(\int_{-1}^2 z dz \right) dx \right) dy =$$

$$= \int_1^2 \left(\int_0^1 x^2 \left[\frac{z^2}{2} \right]_{-1}^2 dx \right) dy = \int_1^2 \left(\frac{3}{2} \int_0^1 x^2 dx \right) dy =$$

$\frac{4}{2} - \frac{1}{2} = \frac{3}{2}$

$$= \frac{3}{2} \left(\int_0^1 x^2 dx \right) \left(\int_1^2 dy \right) = \frac{3}{2} \left[\frac{x^3}{3} \right]_0^1 \left[y \right]_1^2 = \frac{3}{2} \cdot \frac{1}{3} (2-1) = \frac{1}{2}$$

Teda ak zo integrálom je niečo funkcií jednej premennej a mi splnené predpoklady vety o výpočte v našom príklade $f(x,y,z) = x^2 \cdot z = x \cdot 1 \cdot z$

$$\psi_1(x) \cdot \psi_2(y) \cdot \psi_3(z)$$

to je ten trojný integrál niečím jednoduchý integrál

d.j. $\iiint x^2z dx dy dz = \left(\int_0^1 x^2 dx \right) \left(\int_1^2 1 dy \right) \int_{-1}^2 z dz =$

$\langle 0,1 \rangle \times \langle 1,2 \rangle \times \langle -1,2 \rangle$

$$= \left[\frac{x^3}{3} \right]_0^1 \left[y \right]_1^2 \left[\frac{z^2}{2} \right]_{-1}^2 = \frac{1}{3} \cdot 1 \cdot \frac{3}{2} = \frac{1}{2}$$

Podobne vypočítame $\iiint y dx dy dz = \frac{9}{2}$

$\langle 0,1 \rangle \times \langle 1,2 \rangle \times \langle -1,2 \rangle$

Pomocou vety o výpočte ukážeme, že ak pre všetky $(x, y, z) \in E(\bar{a}, \bar{b}) = \langle a_1, b_1 \rangle \times \langle a_2, b_2 \rangle \times \langle a_3, b_3 \rangle$ je $f(x, y, z) = f_1(x) \cdot f_2(y) \cdot f_3(z)$ a sú splnené predpoklady vety o výpočte (napr. ak $f \in \mathbb{R}^3 \times \mathbb{R}$ je spojité na $E(\bar{a}, \bar{b})$) tak

$$\int_{E(\bar{a}, \bar{b})} f(\bar{x}) d\bar{x} = \int_{\langle a_1, b_1 \rangle} \int_{\langle a_2, b_2 \rangle} \int_{\langle a_3, b_3 \rangle} f_1(x) \cdot f_2(y) \cdot f_3(z) dx dy dz =$$

$$= \left(\int_{a_1}^{b_1} f_1(x) dx \right) \left(\int_{a_2}^{b_2} f_2(y) dy \right) \left(\int_{a_3}^{b_3} f_3(z) dz \right)$$

skutočne:

$$\int_{\langle a_1, b_1 \rangle \times \langle a_2, b_2 \rangle \times \langle a_3, b_3 \rangle} f_1(x) f_2(y) f_3(z) dx dy dz = \int_{a_1}^{b_1} \left(\int_{\langle a_2, b_2 \rangle \times \langle a_3, b_3 \rangle} f_1(x) f_2(y) f_3(z) dy dz \right) dx =$$

$$= \int_{a_1}^{b_1} f_1(x) \left(\int_{\langle a_2, b_2 \rangle \times \langle a_3, b_3 \rangle} f_2(y) f_3(z) dy dz \right) dx =$$

$$= \left(\int_{\langle a_2, b_2 \rangle \times \langle a_3, b_3 \rangle} f_2(y) f_3(z) dy dz \right) \left(\int_{a_1}^{b_1} f_1(x) dx \right) =$$

$$= \left(\int_{a_2}^{b_2} \left(\int_{a_3}^{b_3} f_2(y) f_3(z) dz \right) dy \right) \left(\int_{a_1}^{b_1} f_1(x) dx \right) =$$

$$= \left(\int_{a_2}^{b_2} f_2(y) \left(\int_{a_3}^{b_3} f_3(z) dz \right) dy \right) \left(\int_{a_1}^{b_1} f_1(x) dx \right) =$$

$$= \left(\int_{a_3}^{b_3} f_3(z) dz \right) \left(\int_{a_2}^{b_2} f_2(y) dy \right) \left(\int_{a_1}^{b_1} f_1(x) dx \right).$$

$$\bullet \iiint (x+z)e^{x+y} dx dy dz =$$

$$\langle -1, 1 \rangle \times \langle 0, 1 \rangle \times \langle -1, 0 \rangle$$

$$= \iiint x e^x e^y dx dy dz + \iiint z e^x e^y dx dy dz =$$

$$\langle -1, 1 \rangle \times \langle 0, 1 \rangle \times \langle -1, 0 \rangle \quad \langle -1, 1 \rangle \times \langle 0, 1 \rangle \times \langle -1, 0 \rangle$$

$$= \left(\int_{-1}^1 x e^x dx \right) \left(\int_0^1 e^y dy \right) \left(\int_{-1}^0 1 dz \right) + \left(\int_{-1}^1 e^x dx \right) \left(\int_0^1 e^y dy \right) \left(\int_{-1}^0 z dz \right) =$$

$$= \left([x e^x]_{-1}^1 - \int_{-1}^1 e^x dx \right) [e^y]_0^1 [z]_{-1}^0 + [e^x]_{-1}^1 [e^y]_0^1 \left[\frac{z^2}{2} \right]_{-1}^0 =$$

$$= (e + e^{-1} - (e - e^{-1})) (e - 1) \cdot 1 + (e - e^{-1}) (e - 1) \left(-\frac{1}{2}\right) =$$

$$= 2e^{-1}(e - 1) - \frac{1}{2}(e^2 - 1 - e + e^{-1}) =$$

$$= 2 - 2e^{-1} - \frac{1}{2}e^2 + \frac{1}{2} + \frac{1}{2}e - \frac{1}{2}e^{-1} = \frac{5}{2} + \frac{1}{2}e - \frac{1}{2}e^2 - \frac{5}{2}e^{-1}$$

$$\bullet \iiint (x^2 + e^y \sin z) dx dy dz = \iiint x^2 dx dy dz +$$

$$\langle -1, 1 \rangle \times \langle 0, 1 \rangle \times \langle 0, \frac{\pi}{2} \rangle \quad \langle -1, 1 \rangle \times \langle 0, 1 \rangle \times \langle 0, \frac{\pi}{2} \rangle$$

$$+ \iiint e^y \sin z dx dy dz = \left(\int_{-1}^1 x^2 dx \right) \left(\int_0^1 1 dy \right) \left(\int_0^{\frac{\pi}{2}} 1 dz \right) +$$

$$\left(\int_{-1}^1 1 dx \right) \left(\int_0^1 e^y dy \right) \left(\int_0^{\frac{\pi}{2}} \sin z dz \right) = \left[\frac{x^3}{3} \right]_{-1}^1 [y]_0^1 [z]_0^{\frac{\pi}{2}} +$$

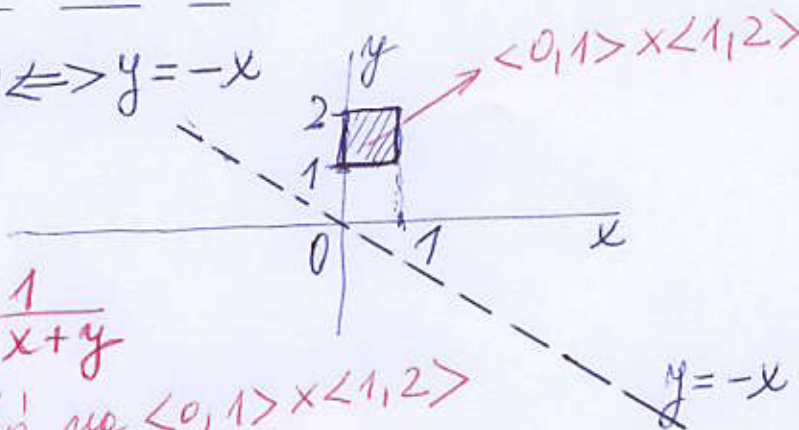
$$+ [x]_{-1}^1 [e^y]_0^1 [-\cos z]_0^{\frac{\pi}{2}} = \left(\frac{1}{3} + \frac{1}{3} \right) (1 - 0) \left(\frac{\pi}{2} - 0 \right) +$$

$$+ (1 + 1) (e - 1) \left(\frac{-\cos \frac{\pi}{2} + \cos 0}{0 + 1 = 1} \right) = \frac{2}{6} \pi + 2(e - 1)$$

$$\checkmark \iiint_{\langle 0,1 \rangle \times \langle 1,2 \rangle \times \langle -1,2 \rangle} \frac{z}{x+y} dx dy dz = \int_{-1}^2 \left(\iint_{\langle 0,1 \rangle \times \langle 1,2 \rangle} z \frac{1}{x+y} dx dy \right) dz = \textcircled{*} \quad \text{5}$$

↓
poderuje dole

pretože: $x+y=0 \Leftrightarrow y=-x$



$$f(x,y) = \frac{1}{x+y}$$

je spojité na $\langle 0,1 \rangle \times \langle 1,2 \rangle$

↓ existuje

teda je každé z pevné zvolené $z \in \langle -1,2 \rangle$:

$$\iint_{\langle 0,1 \rangle \times \langle 1,2 \rangle} z \frac{1}{x+y} dx dy = z \iint_{\langle 0,1 \rangle \times \langle 1,2 \rangle} \frac{1}{x+y} dx dy$$

pre každé $x \in \langle 0,1 \rangle$ existuje

$$\int_1^2 \frac{1}{x+y} dy = [\ln(x+y)]_1^2 = \ln(x+2) - \ln(x+1)$$

$$\textcircled{*} = \int_{-1}^2 z \left(\iint_{\langle 0,1 \rangle \times \langle 1,2 \rangle} \frac{1}{x+y} dx dy \right) dz = \left(\iint_{\langle 0,1 \rangle \times \langle 1,2 \rangle} \frac{1}{x+y} dx dy \right) \cdot \int_{-1}^2 z dz =$$

existuje a je teda = nejaké reálne číslo

$$= \left(\int_0^1 \left(\int_1^2 \frac{1}{x+y} dy \right) dx \right) \cdot \left[\frac{z^2}{2} \right]_{-1}^2 =$$

$\frac{4}{2} - \frac{1}{2} = \frac{3}{2}$

$$= \frac{3}{2} \int_0^1 [\ln(x+y)]_1^2 dx = \frac{3}{2} \int_0^1 (\ln(x+2) - \ln(x+1)) dx$$

$$= \frac{3}{2} \left[(x+2)\ln(x+2) - x - (x+1)\ln(x+1) + x \right]_0^1 = \underline{\underline{3\ln 3 - 4\ln 2}} \cdot \frac{3}{2}$$

✓ $\iint_{\langle 0,1 \rangle \times \langle 1,2 \rangle} \frac{x}{x+y} dx dy = \int_0^1 \left(\int_1^2 \frac{x}{x+y} dy \right) dx =$

$= \int_0^1 \left[x \ln(x+y) \right]_1^2 dx = \int_0^1 \left(\underbrace{x}_{f_1} \underbrace{\ln(x+2)}_{g_1} - \underbrace{x}_{f_2} \underbrace{\ln(x+1)}_{g_2} \right) dx =$

$= \left[\frac{x^2}{2} \ln(x+2) \right]_0^1 - \int_0^1 \frac{x^2}{2} \frac{1}{x+2} dx - \left[\frac{x^2}{2} \ln(x+1) \right]_0^1 +$
par parties

$+ \int_0^1 \frac{x^2}{2} \frac{1}{x+1} dx = \frac{1}{2} \ln 3 - \frac{1}{2} \int_0^1 \frac{(x^2-4)+4}{x+2} dx -$

$- \frac{1}{2} \ln 2 + \frac{1}{2} \int_0^1 \frac{(x^2-1)+1}{x+1} dx = \frac{1}{2} \ln \frac{3}{2} -$

$- \frac{1}{2} \int_0^1 \left(x-2 + \frac{4}{x+2} \right) dx + \frac{1}{2} \int_0^1 \left(x-1 + \frac{1}{x+1} \right) dx =$

$= \frac{1}{2} \ln \frac{3}{2} - \frac{1}{2} \left[\frac{x^2}{2} - 2x + 4 \ln(x+2) \right]_0^1 +$

$+ \frac{1}{2} \left[\frac{x^2}{2} - x + \ln(x+1) \right]_0^1 = \frac{1}{2} \ln \frac{3}{2} - \frac{1}{2} \left(\frac{1}{2} - 2 + \right.$

$\left. + 4 \ln 3 - 4 \ln 2 \right) + \frac{1}{2} \left(\frac{1}{2} - 1 + \ln 2 \right) =$

$= \frac{1}{2} \ln \frac{3}{2} + \frac{3}{4} - 2 \ln \frac{3}{2} - \frac{1}{4} + \frac{1}{2} \ln 2 =$

$= \frac{1}{2} - \frac{3}{2} \ln \frac{3}{2} + \frac{1}{2} \ln 2$
