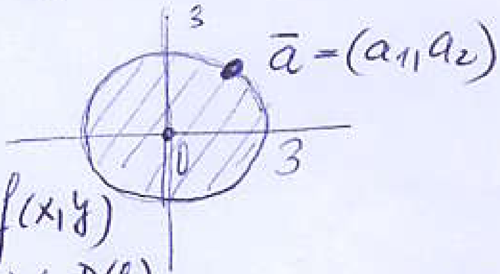
Lokálne extrémy funkcie $f \subseteq R^n \times R$

Ak pre funkciu $f \subseteq R^n_\times R$ existuje $O_\delta(\bar{a}) \subseteq D(f)$ také, že pre všetky $\bar{x} \in O^\circ_\delta(\bar{a})$ je

- $f(\bar{a}) \geq f(\bar{x})$ tak f má v $\bar{a}$ lokálne maximum
- $f(\bar{a}) > f(\bar{x})$ tak f má v $\bar{a}$ ostré lok. maximum
- $f(\bar{a}) \leq f(\bar{x})$ tak f má v $\bar{a}$ lokálne minimum
- $f(\bar{a}) < f(\bar{x})$ tak f má v $\bar{a}$ ostré lok. minimum

pre $\bar{x} \neq \bar{a}$, $\bar{x} \in O^\circ_\delta(\bar{a})$

- Extrém funkcie nemusí byť jej lokálnym extrémom
- $f(x,y) = \sqrt{9 - x^2 - y^2}$ $\qquad D(f) = \{(x,y) \in R^2 \mid x^2 + y^2 \leq 9\}$



napr. pre $\bar{a} = (a_1, a_2)$ také že $a_1^2 + a_2^2 = 9$

je $f(a_1, a_2) = \sqrt{9 - (a_1^2 + a_2^2)} = 0 \leq f(x,y)$ pre všetky $(x,y) \in D(f)$

ale f nemá v $\bar{a}$ lokálny extrém pretože pre žiadne δ neplatí $O_\delta(\bar{a}) \subseteq D(f)$;

teda: $m = 0 = \min f$, ale m nie lok. extrém f.

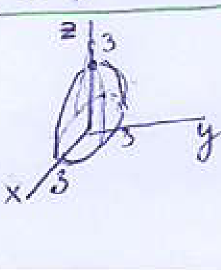
$f(0,0) = \sqrt{9-0-0} = \sqrt{9} = 3 \geq f(x,y)$ pre všetky $(x,y) \in D(f)$

$\Rightarrow$ f má v $\bar{a} = (0,0)$ lokálne maximum

(pretože $\bar{a} = (0,0)$ je vnútorný bod $D(f)$).

okrem toho pre $\bar{x} \neq \bar{a} = (0,0)$ je $f(0,0) > f(x,y)$

teda $f(0,0)$ je ostré lokálne maximum f.

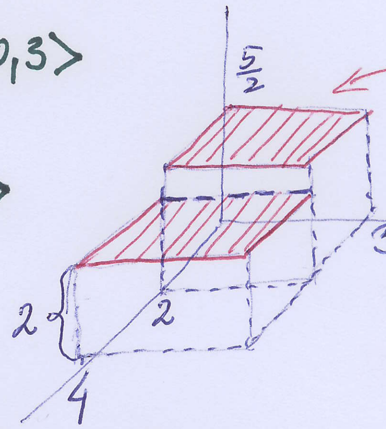


graf f

$H(f) = \langle 0, 3 \rangle$ $\qquad \max f = 3$ $\qquad \min f = 0$

- $f(x,y) = \begin{cases} \frac{5}{2}, & (x,y) \in \langle 0,2\rangle \times \langle 0,3\rangle \\ 2, & (x,y) \in (2,4\rangle \times \langle 0,3\rangle \end{cases}$

(príklad na strane 6)



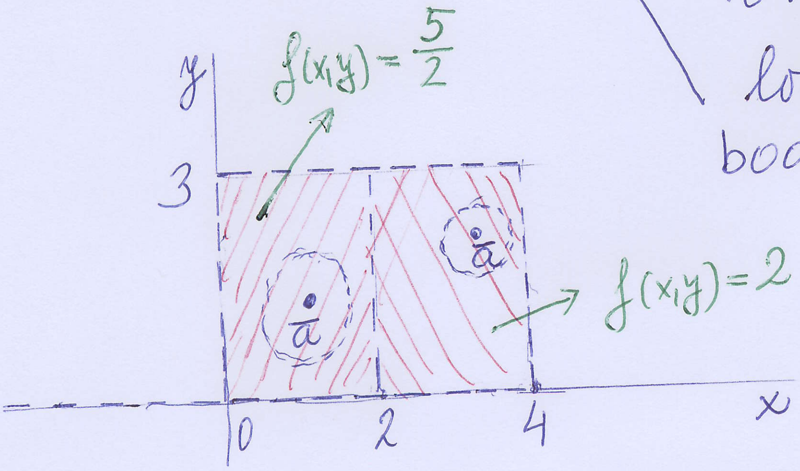
graf f

$\max f = \frac{5}{2}$

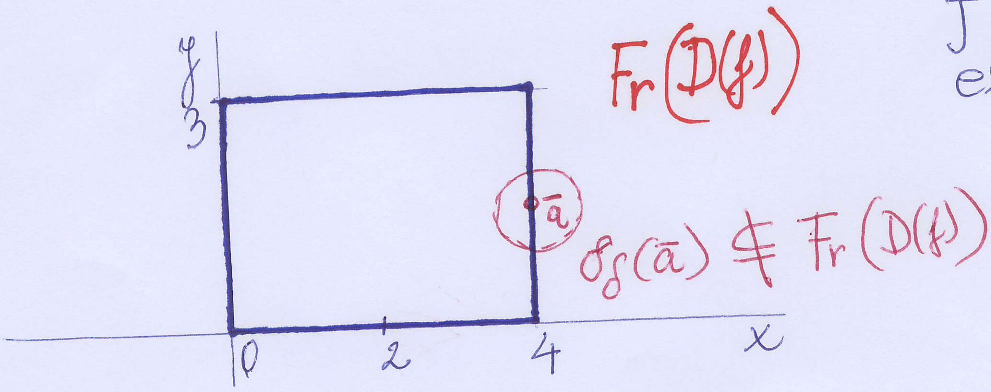
$\min f = 2$

funkcia f má — lokálne minimum v každom bode $\bar{a} \in (2,4) \times (0,3)$: $f(\bar{a}) = 2$

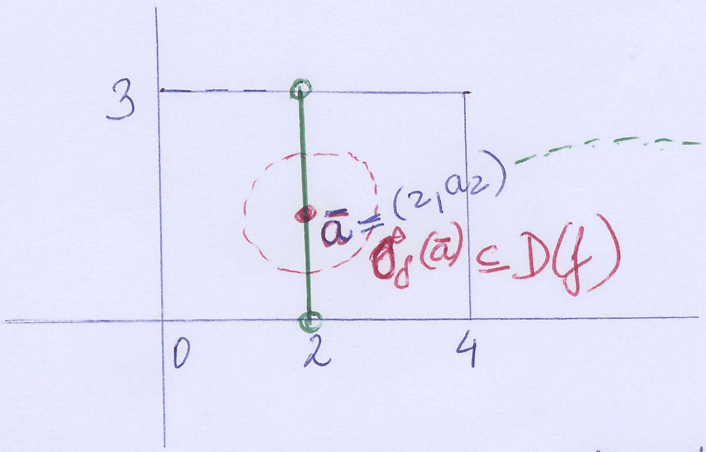
— lokálne maximum v každom bode $\bar{a} \in (0,2) \times (0,3)$: $f(\bar{a}) = \frac{5}{2}$



vnútorné body $D(f)$



f nemá lokálne extrémy v bodoch $\bar{a} \in Fr(D(f))$



f má v bodoch $\bar{a} = (2, a_2)$, pričom $a_2 \in (0,3)$ lokálne maximum $f(\bar{a}) = \frac{5}{2}$ (pretože: pre každé $(x,y) \in O_\delta(\bar{a})$ je $f(x,y) \in \{2, \frac{5}{2}\}$).

zopakujme si:

- Ak funkcia $f \subseteq R^n \times R$ je diferencovateľná v bode $\bar{a}$ a má lokálny extrém v bode $\bar{a}$ potom $\left[\frac{\partial f(\bar{x})}{\partial x_k}\right]_{\bar{x}=\bar{a}} = 0$, $k = 1, \dots, n$ (t.j. $\nabla f(\bar{a}) = \bar{0}$)

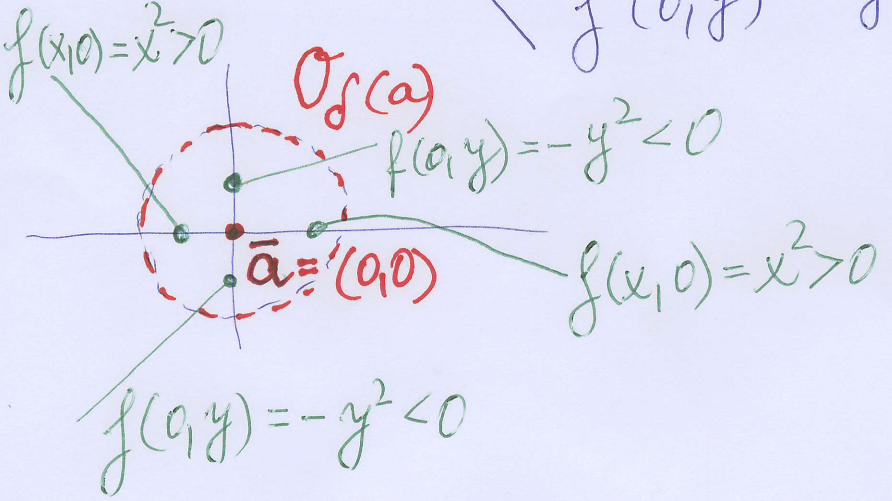
- funkcia nemusí mať lokálny extrem v jej stacionárnom bode

- $f(x,y) = x^2 - y^2 \quad D(f) = R^2$

$\frac{\partial f(x,y)}{\partial x} = 2x$, $\frac{\partial f}{\partial y} = 2y \Rightarrow \bar{a} = (0,0)$ je stacionárny bod f, pretože

$$\left[\frac{\partial f(x,y)}{\partial x}\right]_{\substack{x=0\\y=0}} = [2x]_{\substack{x=0\\y=0}} = 0, \quad \left[\frac{\partial f(x,y)}{\partial y}\right]_{\substack{x=0\\y=0}} = [2y]_{\substack{x=0\\y=0}} = 0.$$

platí však:
- $f(0,0) = 0$
- $f(x,0) = x^2 > 0$ ak $x \neq 0$
- $f(0,y) = -y^2 < 0$ ak $y \neq 0$



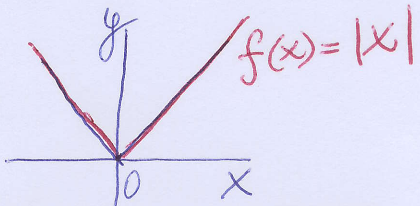
pre každé $O_\delta(a)$ teda pre $(x,y) \in O_\delta(\bar{a})$ je $f(x,y)$
- > 0 ak $y = 0$
- < 0 ak $x = 0$
- $= 0$ ak $x = y = 0$

$\Downarrow$

$f(\bar{a}) = 0$ nie je lokálny extrém f.

- funkcia $f \subseteq R^n \times R$ môže mať lokálny extrém aj v bodoch v ktorých nie je diferencovateľná.

- ak $\underline{n=1}$ $\quad f(x)=|x|$



f má v bode $a=0$ ostré lokálne minimum $f(0)=0$
ale $f'(0)$ neexistuje $\Rightarrow$ f nie je diferenc. v $a=0$
dôkaz je na strane 21

Zopakujme si:

- Parciálne derivácie funkcie $f \subseteq R^n \times R$ vyšších rádov, ktoré sa líšia len v poradí derivovania, ak sú spojité v bode $\bar{a}$, potom sa v bode $\bar{a}$ sebe rovnajú.

- Funkcia $f \subseteq R^n \times R$ je 2-krát diferencovateľná v bode $\bar{a}$ (vnútorný bod $D(f)$) ak f je diferencovateľná na nejakom okolí $O_\delta(\bar{a})$ bodu $\bar{a}$ (t.j. v každom bode $\bar{x} \in O_\delta(\bar{a})$) a jej parciálne derivácie $\frac{\partial f(\bar{x})}{\partial x_k}$ sú diferencovateľné v bode $\bar{a}$. Potom 2-hý diferenciál f v bode $\bar{a}$ je: (symbolicky)

$$\boxed{D^2 f_{\bar{a}}(\bar{x}) = \left[\frac{\partial}{\partial x_1}(x_1-a_1) + \frac{\partial}{\partial x_2}(x_2-a_2) + \dots + \frac{\partial}{\partial x_n}(x_n-a_n)\right]^2 f(\bar{a})}$$

Zrejme ak funkcia $f \subseteq R^n \times R$ má v bode $\bar{a}$ spojité všetky parciálne derivácie 2-hého rádu tak: (1) parciálne derivácie $\frac{\partial f(\bar{x})}{\partial x_k}$ sú diferencov. v $\bar{a}$
(2) parciálne derivácie 2-hého rádu $\left[\frac{\partial^2 f(\bar{a})}{\partial x_j \partial x_k}\right]_{x=\bar{a}} = \left[\frac{\partial^2 f(\bar{a})}{\partial x_k \partial x_j}\right]_{x=\bar{a}}$

Z tvrdení na strane 36 dostávame:

Nech funkcia $f \subseteq R^n \times R$ má na nejakom okolí bodu $\bar{a} = (a_1, a_2, \dots, a_n)$ spojité všetky parciálne derivácie 2-hého rádu, potom

$$D^2 f_{\bar{a}}(\bar{x}) = \sum_{i,j=1}^{n} \left[\frac{\partial^2 f(\bar{x})}{\partial x_j \partial x_i}\right]_{\bar{x}=\bar{a}} (x_i - a_i)(x_j - a_j) \quad \circledast$$

teda pre $n=2$:

$$D^2 f_{\bar{a}}(x,y) = \left[\frac{\partial}{\partial x}(x-a_1) + \frac{\partial}{\partial y}(y-a_2)\right]^2 f(\bar{a}) =$$

$$= \left[\frac{\partial^2 f(x,y)}{\partial x^2}\right]_{\bar{x}=\bar{a}} (x-a_1)^2 + 2\frac{\partial^2 f(x,y)}{\partial y \partial x}(x-a_1)(y-a_2) +$$

$$+ \left[\frac{\partial^2 f(x,y)}{\partial y^2}\right]_{\bar{x}=\bar{a}} (y-a_2)^2 , \quad \text{kde } \bar{a} = (a_1, a_2)$$

pre $n=3$:

$$D^2_{\bar{a}}(x,y,z) = \left[\frac{\partial}{\partial x}(x-a_1) + \frac{\partial}{\partial y}(y-a_2) + \frac{\partial}{\partial z}(z-a_3)\right]^2 f(\bar{a}) =$$

$$= \left[\frac{\partial^2 f(x,y,z)}{\partial x^2}\right]_{\bar{x}=\bar{a}} (x-a_1)^2 + \left[\frac{\partial^2 f(x,y,z)}{\partial y^2}\right]_{\bar{x}=\bar{a}} (y-a_2)^2 +$$

$$+ \left[\frac{\partial^2 f(x,y,z)}{\partial z^2}\right]_{\bar{x}=\bar{a}} (z-a_3)^2 + 2\left[\frac{\partial^2 f(x,y,z)}{\partial y \partial x}\right]_{\bar{x}=\bar{a}} (x-a_1)(y-a_2) +$$

$$+ 2\left[\frac{\partial^2 f(x,y,z)}{\partial z \partial x}\right]_{\bar{x}=\bar{a}} (x-a_1)(z-a_3) + 2\left[\frac{\partial^2 f(x,y,z)}{\partial z \partial y}\right]_{\bar{x}=\bar{a}} (y-a_2)(z-a_3)$$

$D^2 f_{\bar{a}}(\bar{h})$ dostaneme ak položíme $\bar{h} = \bar{x} - \bar{a}$ do $\circledast$ teda $\bar{h} = (h_1, h_2, \dots, h_n) = (x_1 - a_1, x_2 - a_2, \dots, x_n - a_n)$

- $f(x,y,z) = x^2+y^2+z^2-xy+2z+x$; $\bar{a}=\left(-\frac{1}{3},-\frac{2}{3},-1\right)$

$$\frac{\partial f(x,y,z)}{\partial x} = 2x-y+1, \quad \frac{\partial f(x,y,z)}{\partial y} = 2y-x, \quad \frac{\partial f(x,y,z)}{\partial z} = 2z+2$$

$$\frac{\partial^2 f(x,y,z)}{\partial x^2} = 2, \quad \frac{\partial^2 f(x,y,z)}{\partial y^2} = 2, \quad \frac{\partial^2 f(x,y,z)}{\partial z^2} = 2, \quad \frac{\partial^2 f(x,y,z)}{\partial y\,\partial x} =$$

$$= -1 = \frac{\partial^2 f(x,y,z)}{\partial x\,\partial y}, \quad \frac{\partial^2 f(x,y,z)}{\partial z\,\partial x} = 0 = \frac{\partial^2 f(x,y,z)}{\partial x\,\partial z},$$

$$\frac{\partial^2 f(x,y,z)}{\partial z\,\partial y} = 0 = \frac{\partial^2 f(x,y,z)}{\partial y\,\partial z}$$

$$\boxed{D^2 f_{\bar{a}}(\bar{x})} = 2\left(x+\tfrac{1}{3}\right)^2 + 2\left(y+\tfrac{2}{3}\right)^2 + 2(z+1)^2 +$$
$$+ 2(-1)\left(x+\tfrac{1}{3}\right)\left(y+\tfrac{2}{3}\right) + 2\cdot 0\cdot\left(x+\tfrac{1}{3}\right)(z+1) +$$
$$+ 2\cdot 0\cdot\left(y+\tfrac{2}{3}\right)(z+1) =$$
$$= \underline{2\left(x+\tfrac{1}{3}\right)^2 + 2\left(y+\tfrac{2}{3}\right)^2 + 2(z+1)^2 - 2\left(x+\tfrac{1}{3}\right)\left(y+\tfrac{2}{3}\right)}$$

- Ak v predchádzajúcom príklade položíme

$$\bar{x}-\bar{a} = \Big(\underbrace{x+\tfrac{1}{3}}_{h_1}, \underbrace{y+\tfrac{2}{3}}_{h_2}, \underbrace{z+1}_{h_3}\Big) = \bar{h} = (h_1, h_2, h_3)$$

dostaneme

$$D^2 f_{\bar{a}}(\bar{h}) = 2h_1^2 + 2h_2^2 + 2h_3^2 - 2h_1 h_2 =$$
$$= (h_1-h_2)^2 + h_1^2 + h_2^2 + 2h_3^2 > 0$$

pre všetky $\bar{h} \neq (0,0,0)$

- $f(x,y,z) = x^2-y^2-z^2$, $\bar{a} = (a_1,a_2,a_3) \in R^3$

$$\frac{\partial^2 f}{\partial x^2} = 2, \quad \frac{\partial^2 f}{\partial y^2} = -2, \quad \frac{\partial^2 f}{\partial z^2} = -2, \quad \frac{\partial^2 f}{\partial y\partial x} = \frac{\partial^2 f}{\partial z\partial x} = \frac{\partial^2 f}{\partial z\partial y} = 0$$

$$D^2 f_{\bar{a}}(\bar{h}) = 2h_1^2 - 2h_2^2 - 2h_3^2 \quad \begin{cases} > 0 & \text{pre } \bar{h} = (1,0,0) \\ < 0 & \text{pre } \bar{h} = (0,1,0) \end{cases}$$