Neurčitý integrál

Zopakujme si: $\boxed{\int f(x)\,dx = F(x) + C}$ práve

(*) vtedy ak $[F(x)+C]' = f(x)$ na otvorenom intervale J.

Príklad: Ukážte, že platí:

(1) $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C$ na $(-\infty, \infty)$

(2) $\int \dfrac{1}{x}\,dx = \ln x + C$ na $(0, \infty)$

(3) $\int \dfrac{1}{x}\,dx = \ln(-x) + C$ na $(-\infty, 0)$

(2) & (3) $\int \dfrac{1}{x}\,dx = \ln|x| + C$ na $(-\infty, 0), (0, \infty)$

(4) $\int e^x\,dx = e^x + C$ na $(-\infty, \infty)$

(5) $\int \sin x\,dx = -\cos x + C$ na $(-\infty, \infty)$

(6) $\int \cos x\,dx = \sin x + C$ na $(-\infty, \infty)$

(7) $\int \dfrac{1}{\cos^2 x}\,dx = \operatorname{tg} x + C$ na intervaloch
$(2k-1)\frac{\pi}{2} < x < (2k+1)\frac{\pi}{2}$
$k = 0, \pm 1, \pm 2, \dots$

(8) $\int \dfrac{1}{\sin^2 x}\,dx = -\operatorname{cotg} x + C$
na intervaloch
$k\pi < x < (k+1)\pi$
$k = 0, \pm 1, \pm 2, \dots$

(9) $\int \dfrac{1}{1+x^2}\,dx = \begin{cases} \operatorname{arctg} x + C \\ -\operatorname{arccotg} x + C \end{cases}$ na $(-\infty, \infty)$

(10) $\int \frac{1}{1-x^2}\,dx = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + C$

na intervaloch $(-\infty,-1), (-1,1), (1,\infty)$

(11) $\int \frac{1}{\sqrt{1-x^2}}\,dx = \begin{cases} \arcsin x + C & \text{na } (-1,1) \\ -\arccos x + C & \text{na } (-1,1) \end{cases}$

(12) $\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right) + C$

$a \neq 0$ na $(-\infty,\infty)$

(13) $\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$

na ľubovoľnom intervale J na ktorom existuje spojitá derivácia f' (funkcie f) a $f(x) \neq 0$ pre všetky $x \in J$.

(14) $\boxed{\int a^x\,dx = \frac{a^x}{\ln a} + C}$ na $(-\infty,\infty)$

$a > 0,\ a \neq 1$

(15) $\boxed{\int x^a\,dx = \frac{x^{a+1}}{a+1} + C}$ na $(0,\infty)$

$a \in R$ nie je celé číslo

(zopakujme si že pre $a \in R$, ktoré nie je celé číslo je $x^a = e^{a\ln x}$, teda $x \in (0,\infty)$)

(16) $\int \frac{x}{\sqrt{x^2+a^2}}\,dx = \sqrt{x^2+a^2} + C$ na $(-\infty,\infty)$

<u>Dôkaz urobte</u>: derivovaním výsledku získame funkciu pod integrálom. (pozri (*))

(A) $\int \frac{1}{\sqrt{3-3x^2}}\,dx = \frac{1}{\sqrt{3}}\int \frac{1}{\sqrt{1-x^2}}\,dx = \frac{1}{\sqrt{3}} \arcsin x + C$ na $(-1,1)$

(B) $\int \frac{5\cdot 3^x - 2\cdot 4^x}{4^x}\,dx = 5\int \left(\frac{3}{4}\right)^x dx - \int 2\,dx = 5\left(\frac{3}{4}\right)^x \frac{1}{\ln\frac{3}{4}} - 2x + C$

$[3^x]' = [e^{x\ln 3}]' = 3^x \ln 3 \Rightarrow \int 3^x\,dx = \frac{3^x}{\ln 3} + C$ na $(-\infty,\infty)$

(C) $\int \frac{\cos 2x}{\cos^2 x \sin^2 x}\,dx = \int \frac{\cos^2 x - \sin^2 x}{\cos^2 x \sin^2 x}\,dx = \int \frac{1}{\sin^2 x}\,dx - \int \frac{dx}{\cos^2 x} =$

$= -\operatorname{cotg} x - \operatorname{tg} x + C$ napr. na $(0,\frac{\pi}{2}), (\frac{\pi}{2},\pi)$ a pod.

(D) $\int \operatorname{tg}^2 x\,dx = \int \frac{\sin^2 x}{\cos^2 x}\,dx = \int \frac{1-\cos^2 x}{\cos^2 x}\,dx = \int \frac{1}{\cos^2 x}\,dx -$

$-\int dx = \operatorname{tg} x - x + C$ napr. na $(-\frac{\pi}{2},0), (0,\frac{\pi}{2}), (\frac{\pi}{2},\pi)$ a pod.

(E) $\int \operatorname{cotg}^2 x\,dx = \int \frac{\cos^2 x}{\sin^2 x}\,dx = \int \frac{1-\sin^2 x}{\sin^2 x}\,dx =$

$= \int \frac{1}{\sin^2 x}\,dx - \int dx = -\operatorname{cotg} x - x + C$ napr. na $(-\pi,0), (0,\pi)$ a pod.

(F) $\int \frac{dx}{x-3} = \ln|x-3| + C$ na $(-\infty,3), (3,\infty)$

(G) $\int \frac{dx}{x^2+4x+5} = \int \frac{dx}{(x+2)^2+1} = \operatorname{arctg}(x+2) + C$ na $(-\infty,\infty)$

(H) $\int \frac{3x}{x^2+4x+5}\,dx = \frac{3}{2}\int \frac{2x}{x^2+4x+5}\,dx = \frac{3}{2}\int \frac{(2x+4)-4}{x^2+4x+5}\,dx =$

$= \frac{3}{2}\int \frac{2x+4}{x^2+4x+5}\,dx - \frac{12}{2}\int \frac{1}{(x+2)^2+1}\,dx =$

$= \frac{3}{2}\ln(x^2+4x+5) - 6\operatorname{arctg}(x+2) + C$ na $(-\infty,\infty)$

$\boxed{\text{skúška}}$: derivovaním výsledku

Určitý Riemanov integrál

Newton-Leibnizov vzorec

(A) ✓ $\int_0^{\frac{1}{2}} \frac{1}{\sqrt{3-3x^2}}\,dx = \frac{1}{\sqrt{3}} \int_0^{\frac{1}{2}} \frac{1}{\sqrt{1-x^2}}\,dx =$

$= \frac{1}{\sqrt{3}} \left[\arcsin x\right]_0^{\frac{1}{2}} = \frac{1}{\sqrt{3}} \left(\underbrace{\arcsin\frac{1}{2}}_{\frac{\pi}{6}} - \underbrace{\arcsin 0}_{0}\right) =$

$= \frac{1}{\sqrt{3}}\left(\frac{\pi}{6} - 0\right) = \frac{1}{\sqrt{3}} \cdot \frac{\pi}{6}$, pretože $0, \frac{1}{2} \in (-1, 1)$ teda $\langle 0, \frac{1}{2}\rangle \subseteq (-1,1)$

(pozri (A) v neurčitých integráloch)

(D) ✓ $\int_0^{\frac{\pi}{4}} tg^2 x\,dx = \left[tg\, x - x\right]_0^{\frac{\pi}{4}} = \underbrace{tg\frac{\pi}{4}}_{=1} - \frac{\pi}{4} - \underbrace{tg\, 0}_{=0} - 0 =$

$= \underline{1 - \frac{\pi}{4}}$ pretože: $0, \frac{\pi}{4} \in \langle 0, \frac{\pi}{2})$ teda $\langle 0, \frac{\pi}{4}\rangle \subseteq \langle 0, \frac{\pi}{2})$

(pozri (D) v neurč. integráloch).

(F) ✓ $\int_1^2 \frac{dx}{x-3} = \left[\ln|x-3|\right]_1^2 = \underbrace{\ln 1}_{0} - \ln 2 = \underline{\underline{-\ln 2}}$

pretože: $1, 2 \in (-\infty, 3)$ teda $\langle 1, 2\rangle \subseteq (-\infty, 3)$

✓ $\int_5^9 \frac{dx}{x-3} = \left[\ln|x-3|\right]_5^9 = \ln 6 - \ln 2 = \ln\frac{6}{2} =$

$= \underline{\underline{\ln 3}}$, pretože: $5, 9 \in (3, \infty)$ teda $\langle 5, 9\rangle \subseteq (3, \infty)$

(pozri (F) v neurč. integráloch)

(H) ✓ $\int_0^1 \frac{3x}{x^2+4x+5}\,dx = \frac{3}{2}\int_0^1 \frac{2x+4}{x^2+4x+5}\,dx - 6\int_0^1 \frac{1}{(x+2)^2+1}\,dx$

$= \left[\frac{3}{2}\ln(x^2+4x+5) - 6\,\mathrm{arctg}\,(x+2)\right]_0^1 =$

$= \frac{3}{2}\ln 10 - 6\,\mathrm{arctg}\, 3 - \left(\frac{3}{2}\ln 5 - 6\,\mathrm{arctg}\, 2\right) =$

$= \underline{\frac{3}{2}\ln 10 - 6\,\mathrm{arctg}\, 3 - \frac{3}{2}\ln 5 + 6\,\mathrm{arctg}\, 2}$,

(pozri (H) v neurč. int.) pretože $\langle 0, 1\rangle \subseteq (-\infty, \infty)$