(1) (a) Kedy funkcia f nadobúda ($f \subseteq R^n \times R$) v bode $\bar{a} \in R^n$ lokálne maximum (ostré lokálne maximum)?

Ak existuje $O_\delta(\bar{a}) \subseteq D(f)$ taká že pre všetky $\bar{x} \in O_\delta(\bar{a})$ platí $f(\bar{x}) \leq f(\bar{a})$ (2,5)

$(f(\bar{x}) < f(\bar{a}))$ (2,5)

(b) Ako definujeme $\left[\frac{\partial^2 f(\bar{x})}{\partial x_j \cdot \partial x_k}\right]_{\bar{x}=\bar{a}}$?

Ak existuje $\frac{\partial f(\bar{x})}{\partial x_k}$ na nejakom $O_\delta(\bar{a})$ a existuje $\left[\frac{\partial}{\partial x_j}\left(\frac{\partial f(\bar{x})}{\partial x_k}\right)\right]_{\bar{x}=\bar{a}}$ tak ju označujeme $\left[\frac{\partial^2 f(\bar{x})}{\partial x_j \partial x_k}\right]_{\bar{x}=\bar{a}}$ a nazývame 2-hou parc. deriváciu f podľa x_k a x_j v bode $\bar{a}$.

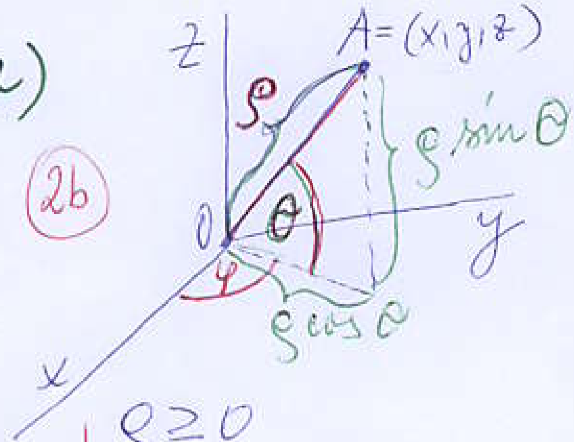
5 bodov

2 (a) Vysvetlite geometrický význam sférických súradníc $(\varrho, \varphi, \theta)$ bodu $(x, y, z) \in R^3$ (aj nakreslite). 5b

(b) Napíšte transformačné rovnice a vetu pre transformáciu trojného integrálu funkcie f pomocou sférických súradníc. 5b

popis ϱ, φ, θ --- 3 body

(a)



2b

ϱ ... vzdialenosť bodu A od začiatku

φ ... uhol spojnice $(x, y, 0)$ so začiatkom a kladného smeru osi x

(teda priemetu (x, y, z) do roviny x, y a kladného smeru osi x)

θ ... uhol spojnice A so začiatkom a rovinou xy

1b
$$\begin{cases} \varrho \geq 0 \\ \varphi \in \langle 0, 2\pi) \\ \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \end{cases}$$

(b) 1b
$$\begin{cases} x = \varrho \cos\varphi \cos\theta \\ y = \varrho \sin\varphi \cos\theta \\ z = \varrho \sin\theta \end{cases}$$

$|J| = \varrho^2 \cos\theta$ 1b

3b $\boxed{\text{Veta}}$. Nech oblasť $A \subseteq R^3$ je vo sfér. súr. popísaná nerovnosťami:

$$\left.\begin{array}{l} \left.\begin{array}{l} \alpha \leq \varphi \leq \beta \\ h_1(\varphi) \leq \varrho \leq h_2(\varphi) \end{array}\right\} D \\ F_1(\varrho, \varphi) \leq \theta \leq F_2(\varrho, \varphi) \end{array}\right\} \begin{array}{l} h_1, h_2 \text{ spojité na } \langle \alpha, \beta \rangle \\ F_1, F_2 \text{ spojité na } D \end{array}$$

Nech f je spojitá na A, potom

$$\iiint_A f(x, y, z)\, dx\, dy\, dz = \int_\alpha^\beta \left(\int_{h_1(\varphi)}^{h_2(\varphi)} \left(\int_{F_1(\varrho, \varphi)}^{F_2(\varrho, \varphi)} f(\varrho\cos\varphi\cos\theta, \varrho\sin\varphi\sin\theta, \varrho\sin\theta)\, \varrho^2\cos\theta\, d\theta \right) d\varrho \right) d\varphi$$

3) Vypočítajte a urobte skúšku:

(a) $\int \frac{4x}{x^2-6x+13}dx = 2\int \frac{2x-6+6}{x^2-6x+13}dx =$

$= 2\int \frac{2x-6}{x^2-6x+13}dx + 12\int \frac{dx}{(x-3)^2+4} =$

$= 2\ln(x^2-6x+13) + 12\cdot\frac{1}{2}\,\mathrm{arctg}\,\frac{x-3}{2} + C =$

$= 2\ln(x^2-6x+13) + 6\,\mathrm{arctg}\,\frac{x-3}{2} + C$ (4b)

skúška: $\left[2\ln(x^2-6x+13) + 6\,\mathrm{arctg}\,\frac{x-3}{2} + C\right]' =$

$= 2\,\frac{2x-6}{x^2-6x+13} + 6\,\frac{1}{2}\,\frac{1}{1+\frac{(x-3)^2}{4}} =$

$= \frac{4x-12}{x^2-6x+13} + \frac{3\cdot 4}{4+x^2-6x+9} = \frac{4x}{x^2-6x+13}$ (1b)

(b) $\int (x-3)\ln x\,dx = \frac{(x-3)^2}{2}\ln x - \int \frac{(x-3)^2}{2}\cdot\frac{1}{x}dx =$

$= \frac{(x-3)^2}{2}\ln x - \frac{1}{2}\int \frac{x^2-6x+9}{x}dx =$

$= \frac{(x-3)^2}{2}\ln x - \frac{1}{2}\int \left(x-6+\frac{9}{x}\right)dx =$

$= \frac{(x-3)^2}{2}\ln x - \frac{1}{2}\left(\frac{x^2}{2} - 6x + 9\ln x\right) + C$ (4b)

skúška: $\left[\frac{(x-3)^2}{2}\ln x - \frac{1}{2}\left(\frac{x^2}{2} - 6x + 9\ln x\right) + C\right]' =$

$= \frac{\cancel{2}(x-3)}{\cancel{2}}\ln x + \frac{1}{2}(x-3)^2\cdot\frac{1}{x} - \frac{1}{2}\left(x-6+\frac{9}{x}\right) =$

$= (x-3)\ln x + \cancel{\frac{1}{2}\,\frac{x^2-6x+9}{x}} - \cancel{\frac{1}{2}\,\frac{x^2-6x+9}{x}} =$

$= (x-3)\ln x$ (1b)

(4) (a) Zistite či existuje (resp. vypoči):

$$\lim_{\substack{x \to 0 \\ y \to 1}} \frac{x^2+y^2}{xy}$$

(a) $\left(\frac{1}{k}, \frac{k+1}{k}\right) \xrightarrow[k\to\infty]{} (0,1)$

$$f\left(\frac{1}{k}, \frac{k+1}{k}\right) = \frac{\frac{1}{k^2} + \frac{(k+1)^2}{k^2}}{\frac{1}{k}\cdot\frac{k+1}{k}} = \frac{1+(k+1)^2}{k+1} =$$

5b

$$= \frac{1}{k+1} + k+1 \xrightarrow[k\to\infty]{} \infty$$ a teda táto limita neexistuje

(b) Vypočítajte $\left[\frac{\partial f(x,y)}{\partial x}\right]_{\substack{x=1\\y=1}}$, $\left[\frac{\partial f(x,y)}{\partial y}\right]_{\substack{x=1\\y=1}}$

ak $f(x,y) = \frac{x^2+y^2}{xy}$

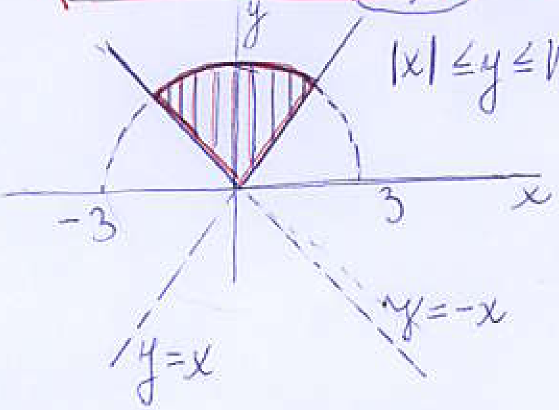
2,5 $$\left[\frac{\partial f(x,y)}{\partial x}\right]_{\substack{x=1\\y=1}} = \left[\frac{2x\,xy-(x^2+y^2)y}{x^2y^2}\right]_{\substack{x=1\\y=1}} = \left[\frac{x^2y-y^3}{x^2y^2}\right]_{\substack{x=1\\y=1}} = \frac{0}{1} = 0$$

2,5 $$\left[\frac{\partial f(x,y)}{\partial y}\right]_{\substack{x=1\\y=1}} = \left[\frac{2y\,xy-(x^2+y^2)x}{x^2y^2}\right]_{\substack{x=1\\y=1}} = \left[\frac{y^2x-x^3}{x^2y^2}\right]_{\substack{x=1\\y=1}} = \frac{0}{1} = 0$$

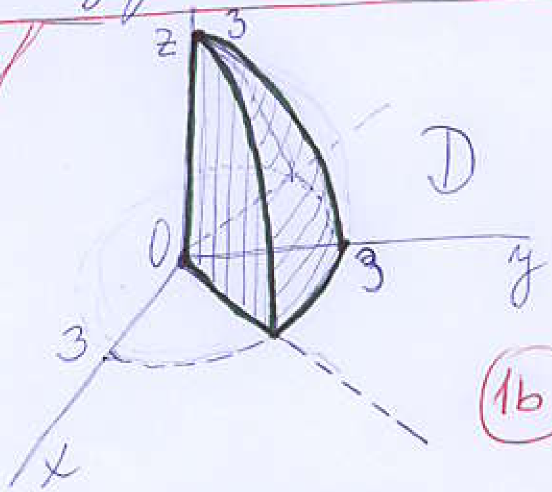
(5) Vypočítajte $\iiint\limits_D z(x^2+y^2)\,dx\,dy\,dz$, kde $D\subseteq R^3$ je dané nerovnosťami: $|x|\le y\le\sqrt{9-x^2}$

$0\le z\le\sqrt{9-x^2-y^2}$

Nakreslite D v R^3 a jeho kolmý priemet do roviny xy ($|x|\le y\le\sqrt{9-x^2}$).



1b



1b

transf.
$$\left.\begin{aligned} x&=\varrho\cos\theta\cos\varphi\\ y&=\varrho\cos\theta\sin\varphi\\ z&=\varrho\sin\theta\end{aligned}\right\}\ |J|=\varrho^2\cos\theta$$

$$0\le\varrho\le 3$$
$$\frac{\pi}{4}\le\varphi\le\frac{3\pi}{4}$$
$$0\le\theta\le\frac{\pi}{2}$$

2b

$$\iiint\limits_D z(x^2+y^2)\,dx\,dy\,dz=\int_0^{\frac{\pi}{2}}\left(\int_0^3\left(\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\underbrace{\varrho\sin\theta}_{z}\cdot\underbrace{\varrho^2\cos^2\theta}_{x^2+y^2}\cdot\underbrace{\varrho^2\cos\theta}_{|J|}\,d\varphi\right)d\varrho\right)d\theta=$$

$$=\left(\int_0^3\varrho^5\,d\varrho\right)\left(\int_0^{\frac{\pi}{2}}\cos^3\theta\sin\theta\,d\theta\right)\left(\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}d\varphi\right)=$$

$$=\left[\frac{\varrho^6}{6}\right]_0^3\left[\varphi\right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\cdot\int_1^0 u^3(-1)\,du=\frac{3^6}{6}\cdot\frac{\pi}{2}\cdot\left[\frac{u^4}{4}\right]_0^1=\frac{3^6\pi}{48}=$$

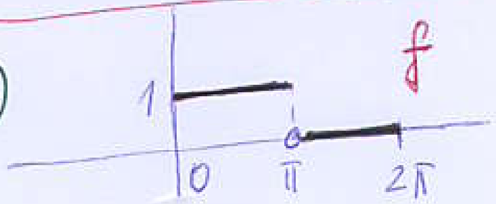
$$=\boxed{\frac{3^5\pi}{16}}$$

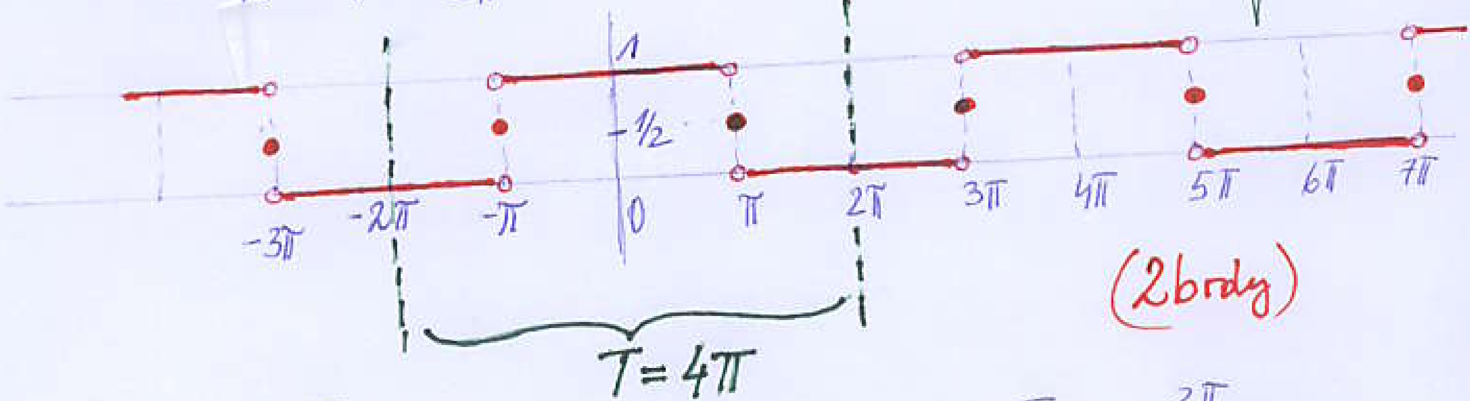
6 bodov

subst.: $u=\cos\theta$
$du=-\sin\theta\,d\theta$

6) Pre funkciu $f(x) = \begin{cases} 1, & x \in \langle 0, \pi \rangle \\ 0, & x \in (\pi, 2\pi \rangle \end{cases}$ (a) nakreslite párne periodické pokračovanie, (b) napíšte kosínusový rad pre f a $\langle 0, 2\pi \rangle$.

(a)





(2 body)

$$a_0 = \frac{4}{T}\int_0^{\frac{T}{2}} f(x)\,dx = \frac{4}{4\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\left(\int_0^{\pi} 1\,dx + \int_{\pi}^{2\pi} 0\,dx\right) =$$

$$= \frac{1}{\pi}\left[x\right]_0^{\pi} = \frac{1}{\pi}(\pi - 0) = \underline{1}$$

(2 body)

$$a_n = \frac{4}{T}\int_0^{\frac{T}{2}} f(x)\cos\frac{2\pi n x}{T}\,dx = \frac{4}{4\pi}\int_0^{2\pi} f(x)\cos\frac{2\pi n x}{4\pi}\,dx =$$

$$= \frac{1}{\pi}\left(\int_0^{\pi}\cos\frac{nx}{2}\,dx + \int_{\pi}^{2\pi} 0\,dx\right) =$$

$$= \frac{1}{\pi}\left[\frac{2}{n}\sin\frac{nx}{2}\right]_0^{\pi} = \underline{\frac{2}{n\pi}\sin\frac{n\pi}{2}}$$

(2 body)

$$\bar{f}(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi}{2}\cos\frac{nx}{2}$$

(4 body)