(1) Kedy funkciu $f \subseteq R^3 \times R$ nazývame diferencovateľnou v bode $\bar{a} = (a_1, a_2, a_3) \in R^3$, čo nazývame jej diferenciálom v bode $\bar{a}$ a čo gradientom v bode $\bar{a}$ a aký je ich vzťah a geometrický význam?

$f \subseteq R^3 \times R$ je diferencovateľná v bode $\bar{a} = (a_1, a_2, a_3)$ ak existujú $\left[\frac{\partial f(x,y,z)}{\partial x_k}\right]_{\bar{x}=\bar{a}}$ a funkcie ε_k $(k = 1, 2, 3) \subseteq R^3 \times R$ také že pre všetky $\bar{x} = (x, y, z) \in O_\delta(\bar{a})$ platí:

$$f(x,y,z) - f(a_1,a_2,a_3) = \left[\frac{\partial f(\bar{x})}{\partial x}\right]_{\bar{x}=\bar{a}}(x-a_1) + \left[\frac{\partial f(\bar{x})}{\partial y}\right]_{\bar{x}=\bar{a}}(y-a_2) + \left[\frac{\partial f(\bar{x})}{\partial z}\right]_{\bar{x}=\bar{a}}(z-a_3) + \varepsilon_1(\bar{x})(x-a_1) + \varepsilon_2(\bar{x})(y-a_2) + \varepsilon_3(\bar{x})(z-a_3)$$

5b

diferenciál $$Df_{\bar{a}}(\bar{x}) = \left[\frac{\partial f}{\partial x}\right]_{\bar{x}=\bar{a}}(x-a_1) + \left[\frac{\partial f}{\partial y}\right]_{\bar{x}=\bar{a}}(y-a_2) + \left[\frac{\partial f}{\partial z}\right]_{\bar{z}=\bar{a}}(z-a_3)$$ 1b

gradient $$\nabla f(\bar{a}) = \left(\left[\frac{\partial f}{\partial x}\right]_{\bar{x}=\bar{a}}, \left[\frac{\partial f}{\partial y}\right]_{\bar{x}=\bar{a}}, \left[\frac{\partial f}{\partial z}\right]_{\bar{z}=\bar{a}}\right)$$ 1b

teda $Df_{\bar{a}}(x) = \nabla f(\bar{a}) \cdot (\bar{x} - \bar{a})$ 1b

pričom rovnica dotykovej roviny ku ploche $f(x,y,z) = c$ v bode $\bar{a} = (a_1, a_2, a_3)$ je

$$Df_{\bar{a}}(\bar{x}) = 0$$

resp. $$\nabla f(\bar{a}) \cdot (\bar{x} - \bar{a}) = 0$$ 1b

1b

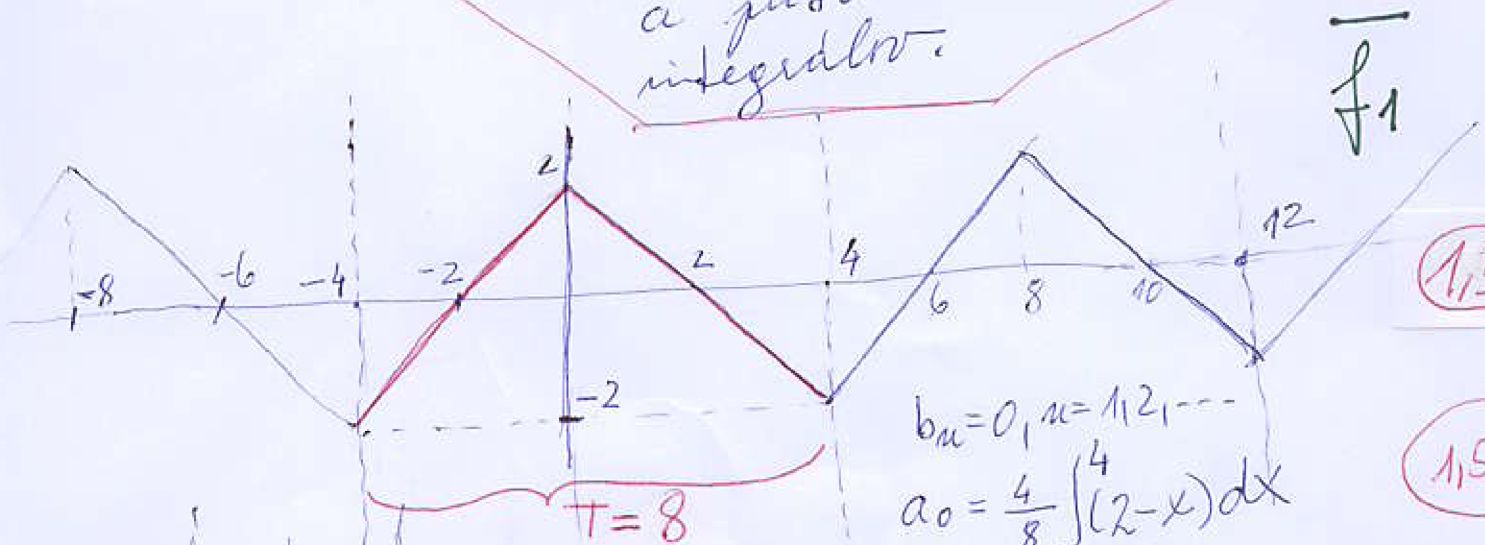
$\nabla f(\bar{a})$ je vektor normály tej dotykovej roviny

② (a) Ako definujeme normalizované periodické pokračovanie $\bar{f}$ po čiastkovo spojitej funkcie f na intervale $\langle a, a+T\rangle$, $T>0$. ?

4 b

$$\bar{f}(x) = \begin{cases} \frac{1}{2}\left\{\lim_{x\to a+} f(x) + \lim_{x\to a-} f(x)\right\} & \text{pre } x=a \\ \frac{1}{2}\left\{\lim_{t\to x+} f(t) + \lim_{t\to x-} f(t)\right\} & \text{pre } x\in(a, a+T) \\ \bar{f}(x+T) & \text{pre každé } x\in R \end{cases}$$

(b) Pre funkciu $f(x)=2-x$ na $\langle 0,4\rangle$ nakreslite **(b1)** párne periodické pokračovanie $\bar{f}_1$ a vyjadrite Fourrierove koeficienty integrálom
(b2) nepárne per. pokračovanie $\bar{f}_2$ a príslušné Fourr. koef. v tvare integrálov.



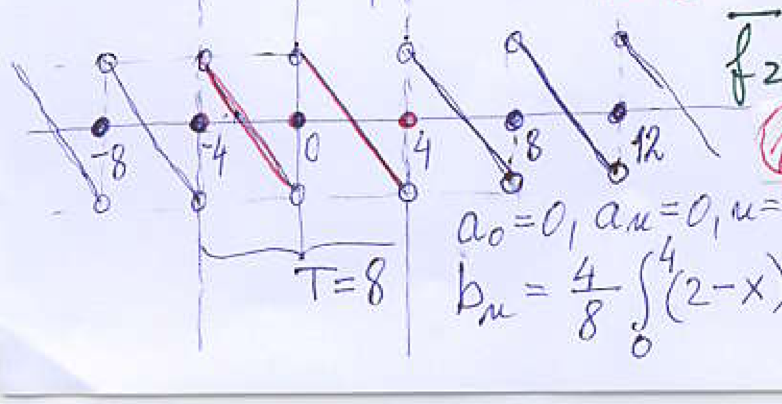
1,5 b

$b_n = 0$, $n = 1, 2, \dots$

$$a_0 = \frac{4}{8}\int_0^4 (2-x)\,dx$$

$$a_n = \frac{4}{8}\int_0^4 (2-x)\cos\frac{2\pi n x}{8}\,dx$$

1,5 b



1,5 b

$a_0 = 0$, $a_n = 0$, $n = 1, 2, \dots$

$$b_n = \frac{4}{8}\int_0^4 (2-x)\sin\frac{2\pi n x}{8}\,dx$$

1,5 b

3) Vypočítajte

(a) $\int \frac{3x-1}{x^2+4x+8}\,dx$ (4b) (b) $\int x \operatorname{arctg} x\, dx$ (4b) a urobte skúšku! (1b) + (1b)

$$\int \frac{3x-1}{x^2+4x+8}\,dx = \frac{3}{2}\int \frac{2x-\frac{2}{3}+4-4}{x^2+4x+8}\,dx = \frac{3}{2}\int \frac{2x+4}{x^2+4x+8}\,dx -$$

$$+\frac{3}{2}\left(-\frac{14}{3}\right)\int \frac{1}{(x+2)^2+4}\,dx = \frac{3}{2}\ln(x^2+4x+8) -$$

$$-7\cdot\frac{1}{2}\operatorname{arctg}\frac{x+2}{2} + C \quad (4b)$$

skúška: $\left[\frac{3}{2}\ln(x^2+4x+8) - \frac{7}{2}\operatorname{arctg}\frac{x+2}{2} + C\right]' =$

$$= \frac{3}{2}\,\frac{2x+4}{x^2+4x+8} - 7\cdot\frac{1}{2}\,\frac{1}{1+\frac{(x+2)^2}{4}}\cdot\frac{1}{2} = \quad (1b)$$

$$= \frac{3x+6}{x^2+4x+8} - \frac{7}{4+(x+2)^2} = \frac{3x-1}{x^2+4x+8}, \text{ č.b.d.}$$

$$\int \underbrace{x}_{u'}\,\underbrace{\operatorname{arctg} x}_{v}\,dx = \frac{x^2}{2}\operatorname{arctg} x - \int \frac{x^2}{2}\,\frac{1}{x^2+1}\,dx =$$

$$= \frac{x^2}{2}\operatorname{arctg} x - \frac{1}{2}\left\{\int \frac{x^2+1}{x^2+1}\,dx - \int \frac{1}{x^2+1}\,dx\right\} =$$

$$= \frac{x^2}{2}\operatorname{arctg} x - \frac{1}{2}x + \frac{1}{2}\operatorname{arctg} x + C \quad (4b)$$

skúška: $\left(\frac{x^2}{2}\operatorname{arctg} x - \frac{1}{2}x + \frac{1}{2}\operatorname{arctg} x\right)' =$

$$= x\operatorname{arctg} x + \frac{x^2}{2}\,\frac{1}{1+x^2} - \frac{1}{2} + \frac{1}{2}\,\frac{1}{1+x^2} =$$

$$= x\operatorname{arctg} x + \frac{1}{2}\,\frac{x^2-\cancel{x^2}-1+1}{1+x^2} = x\operatorname{arctg} x, \text{ č.b.d.} \quad (1b)$$

4 Pre funkciu $f(x,y) = \begin{cases} \dfrac{x^3-y^3+2(x^2+y^2)}{x^2+y^2}, & (x,y) \neq (0,0) \\ b, & (x,y) = (0,0) \end{cases}$

zistite (a) či existuje $b \in R$ také aby f bola spojitá v bode $\bar{a} = (0,0)$ 5b

(b) či existuje (resp. vypočítajte) $\left[\dfrac{\partial f(x,y)}{\partial x}\right]_{\substack{x=0 \\ y=0}}$ 5b

---

(a) $$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^3-y^3+2(x^2+y^2)}{x^2+y^2} = \lim_{\substack{x \to 0 \\ y \to 0}} \left(\frac{x^3-y^3}{x^2+y^2} + 2\right) =$$

$$= \lim_{\substack{x \to 0 \\ y \to 0}} \Bigg(\underbrace{x}_{\to 0} \underbrace{\frac{x^2}{x^2+y^2}}_{\substack{\in[0,1] \\ \text{pre } (x,y)\neq(0,0)}} - \underbrace{y}_{\to 0} \underbrace{\frac{y^2}{x^2+y^2}}_{\substack{\in[0,1] \\ \text{pre } (x,y)\neq(0,0)}} + 2\Bigg) = 0 - 0 + 2 = \boxed{2}$$ 4

pre $b = 2$ je f spojitá v $\bar{a} = (0,0)$ 1

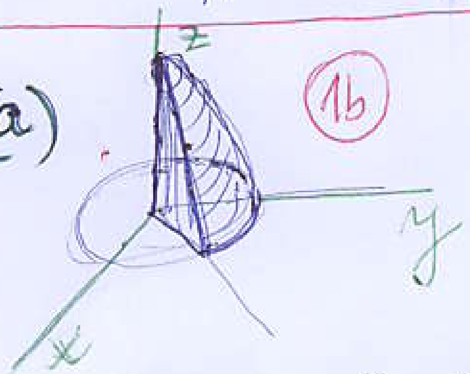
(b) $$\left[\frac{\partial f(x,y)}{\partial x}\right]_{\substack{x=0 \\ y=0}} = \lim_{x \to 0} \frac{f(x,0) - f(0,0)}{x - 0} =$$ 1

$$= \lim_{x \to 0} \frac{\frac{x^3+2x^2}{x^2} - 2}{x - 0} = \lim_{x \to 0} \frac{x+2-2}{x} = \lim_{x \to 0} \frac{x}{x} =$$

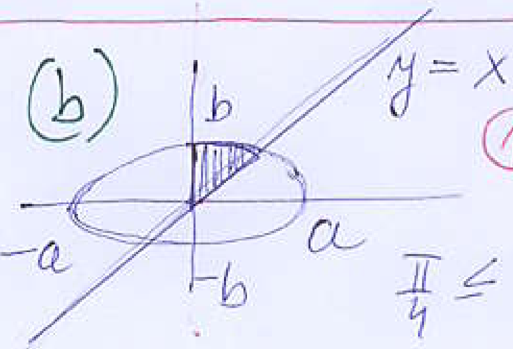
$$= \lim_{x \to 0} 1 = 1$$ 4

pretože pre $x \neq 0$ je $\frac{x}{x} = 1$

(5) Oblasť $A \subseteq R^3$ je ohraničená elipsoidom $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ a rovinami $x=0, y\geq 0, y-x=0, z\geq 0$
(a) nakreslite A, (b) nakreslite priemet A do roviny xy, (c) popíšte A použitím modifikovanej transformácie pomocou sférických súradníc, (d) vypočítajte C(A), (e) vypočítajte $\iiint_A z\,dx\,dy\,dz$.

(a) 

(1b)

(b) 

(1b)

$\frac{\pi}{4} \leq \varphi \leq \frac{\pi}{2}$

(c) $\left.\begin{array}{l} x = a\rho\cos\varphi\cos\theta \\ y = b\rho\sin\varphi\cos\theta \\ z = c\rho\sin\theta \end{array}\right\} |J| = abc\rho^2\cos\theta$ (1b)

$$A: \begin{cases} \frac{\pi}{4} \leq \varphi \leq \frac{\pi}{2} \\ 0 \leq \rho \leq 1 \\ 0 \leq \theta \leq \frac{\pi}{2} \end{cases}$$ (1b)

(d) $C(A) = \iiint_A dx\,dy\,dz = abc\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(\int_0^1\left(\int_0^{\frac{\pi}{2}}\rho^2\cos\theta\,d\theta\right)d\rho\right)d\varphi =$

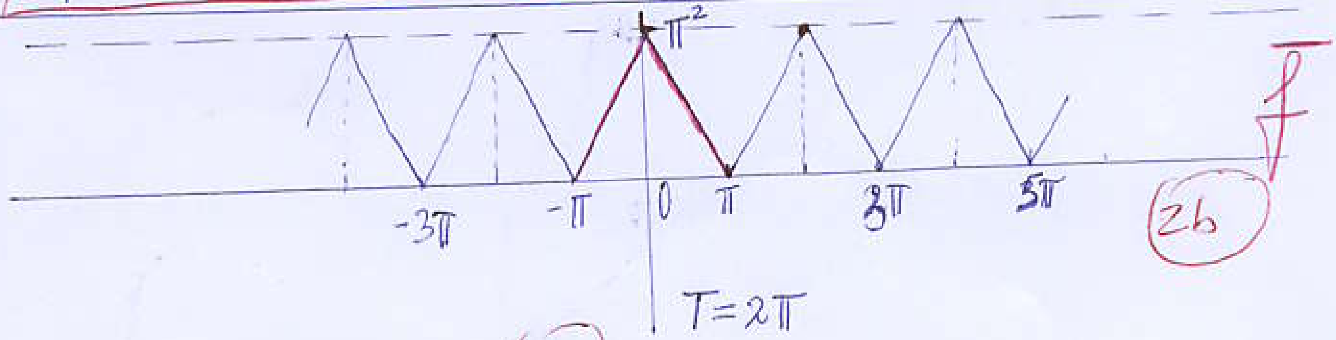
$= abc\,[\varphi]_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left[\frac{\rho^3}{3}\right]_0^1[\sin\theta]_0^{\frac{\pi}{2}} = \frac{\pi}{4}\cdot\frac{1}{3}\cdot abc = abc\frac{\pi}{12}$ (3b)

(e) $\iiint_A z\,dx\,dy\,dz = abc^2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(\int_0^1\left(\int_0^{\frac{\pi}{2}}\rho\sin\theta\,\rho^2\cos\theta\,d\theta\right)d\rho\right)d\varphi =$ (3b)

$= abc^2\left[\frac{\rho^4}{4}\right]_0^1[\varphi]_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left[(-1)\frac{\cos 2\theta}{4}\right]_0^{\frac{\pi}{2}} = abc^2\frac{1}{4}\cdot\frac{\pi}{4}\left(-\frac{1}{4}\right)(-2) = abc^2\frac{\pi}{32}$

6

Napíšte kosínusový rad funkcie $f(x) = \pi^2 - \pi x$ pre interval $\langle 0, \pi \rangle$ a nakreslite graf jeho súčtu.



2b

1b

$b_n = 0, \ n = 1, 2, \dots$

$$a_0 = \frac{4}{T}\int_0^{\frac{T}{2}} f(x)\,dx = \frac{4}{2\pi}\int_0^{\pi}(\pi^2 - \pi x)\,dx = \frac{2}{\pi}\left[\pi^2 x - \pi\frac{x^2}{2}\right]_0^{\pi} =$$

$$= \frac{2}{\pi}\left(\pi^3 - \frac{\pi^3}{2}\right) = \frac{2}{\pi}\cdot\frac{\pi^3}{2} = \pi^2$$

1b

$$a_n = \frac{4}{T}\int_0^{\frac{T}{2}} f(x)\cos\frac{2\pi n x}{2\pi}\,dx = \frac{4}{2\pi}\int_0^{\pi}(\pi^2 - \pi x)\cos nx\,dx =$$

$$= \frac{2}{\pi}\left\{\int_0^{\pi}\pi^2\cos nx\,dx - \pi\int_0^{\pi}\underset{u}{x}\,\underset{v'}{\cos nx}\,dx\right\} =$$

$$= \frac{2}{\pi}\pi^2\underbrace{\left[\frac{\sin nx}{n}\right]_0^{\pi}}_{=0} - 2\left\{\underbrace{\left[x\frac{\sin nx}{n}\right]_0^{\pi}}_{=0} - \int_0^{\pi}\frac{\sin nx}{n}\,dx\right\} =$$

$$= -2\left[\frac{\cos nx}{n^2}\right]_0^{\pi} = 2\left[-\frac{\cos nx}{n^2}\right]_0^{\pi} = 2(-1)\left(\frac{(-1)^n - 1}{n^2}\right)$$

$$= 2\,\frac{(-1)^{n+1} + 1}{n^2}$$

5b

$$\boxed{\bar{f}(x) = \frac{\pi^2}{2} + 2\sum_{n=1}^{\infty}\frac{1 + (-1)^{n+1}}{n^2}\cos nx}$$

1b