Skúška z M2 riešenie

19. mája 2021

Ako sme uviedli v pokynoch ku skúške, príklady hodnotíme maximálnym počtom bodov ak sú napísané logicky, bez preskakovania akýchkoľvek krokov (napríklad derivácia sa objaví bez počítania a ešte aj upravená a podobne).

1. Vypočítajte integrál $\int \arcsin\left(\sqrt{\frac{x}{x+1}}\right) dx$. (**20b**)

$$\int \arcsin\left(\sqrt{\frac{x}{x+1}}\right) dx = \left| \begin{array}{ll} f(x) = \arcsin\left(\sqrt{\frac{x}{x+1}}\right) & g'(x) = 1 \\ f'(x) = \frac{1}{\sqrt{1-\frac{x}{x+1}}}\frac{1}{2\sqrt{\frac{x}{x+1}}}\frac{x+1-x}{(x+1)^2} = \frac{1}{\sqrt{\frac{1}{x+1}}}\frac{1}{2\sqrt{\frac{x}{x+1}}}\frac{1}{(x+1)^2} = \frac{1}{2\sqrt{x}(x+1)} & g(x) = x \end{array} \right|$$

$$= x\arcsin\sqrt{\frac{x}{x+1}} - \int \frac{\sqrt{x}}{2(x+1)}dx = \left| \begin{array}{ll} t = \sqrt{x} & dx = 2tdt \\ x = t^2 & \end{array} \right| =$$

$$= x\arcsin\sqrt{\frac{x}{x+1}} - \int \frac{t}{2(t^2+1)}2tdt = x\arcsin\sqrt{\frac{x}{x+1}} - \int \frac{t^2+1-1}{t^2+1}dt =$$

$$= x\arcsin\sqrt{\frac{x}{x+1}} - \int 1dt + \int \frac{1}{t^2+1}dt = x\arcsin\sqrt{\frac{x}{x+1}} - t + \operatorname{arctg} t + C =$$

$$= x\arcsin\sqrt{\frac{x}{x+1}} - \sqrt{x} + \operatorname{arctg}(\sqrt{x}) + C$$

2a. Pre funkciu $f(z) = (\operatorname{Re} z)^2 + i(\operatorname{Im} z)^2$. **a.** zistite definičný obor $D(f)$, v ktorých bodoch definičného oboru existuje derivácia funkcie f, **b.** vypočítajte $f'(z)$, **c.** vyšetrite, kde je f analytická. (**8b**)

a. $u, v : \boldsymbol{R}^2 \longrightarrow \boldsymbol{R}$, teda $f : \boldsymbol{C} \longrightarrow \boldsymbol{C}$,

$\frac{\partial u}{\partial x} = 2x, \frac{\partial u}{\partial y} = 0, \frac{\partial v}{\partial x} = 0, \frac{\partial v}{\partial y} = 2y \Rightarrow \begin{array}{c} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{array}$ $CRR : 2x = 2y \wedge 0 = 0 \Rightarrow$

$\Rightarrow x = y \Rightarrow f'$ existuje len v bodoch $\{z : \operatorname{Re} z = \operatorname{Im} z\}$.

b. $f'(z) = f'(x+ix) = \frac{\partial u}{\partial x}(x,x) + i\frac{\partial v}{\partial x}(x,x) = 2x + i0 = 2x = 2\operatorname{Re} z = 2\operatorname{Im} z$.

c. f nie je analytická v žiadnom bode, pretože žiadny bod, v ktorom derivácia existuje nemá také okolie aby v ňom derivácia existovala.

2b. Vypočítajte analytickú funkciu $f(z) = f(x+iy) = u(x,y) + iv(x,y)$, ak je daná $v(x,y) = -\frac{y}{x^2+y^2} + 2xy$, pričom $f(1) = 0$ (**12b**)

$v : \boldsymbol{R}^2 \setminus \{(0,0)\} \longrightarrow \boldsymbol{R}$, teda $f : \boldsymbol{C} \setminus \{0\} \longrightarrow \boldsymbol{C}$,

$$\left. \begin{array}{r} \frac{\partial v}{\partial x} = -y\frac{\partial}{\partial x}\left(\frac{1}{x^2+y^2}\right) + 2y = -y\left(-\frac{2x}{(x^2+y^2)^2}\right) + 2y = \frac{2xy}{(x^2+y^2)^2} + 2y \\ \frac{\partial v}{\partial y} = -\frac{1(x^2+y^2) - y2y}{(x^2+y^2)^2} + 2x = \frac{y^2-x^2}{(x^2+y^2)^2} + 2x \end{array} \right\} \Rightarrow$$

f je analytická $\Rightarrow$

$\Rightarrow$ platia C-R rovnice, $\exists u(x,y) :$ $\begin{array}{c} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{array}$

$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow \frac{\partial u}{\partial y} = -\frac{2xy}{(x^2+y^2)^2} - 2y \Rightarrow u(x,y) = \int \left(-\frac{2xy}{(x^2+y^2)^2} - 2y \right) dy + \Phi(x) =$

$= -x \int \frac{2y}{(x^2+y^2)^2} dy - y^2 + \Phi(x) = \left| \begin{array}{l} t = x^2+y^2 \\ dt = 2ydy \end{array} \right| = -x\left(-\frac{1}{x^2+y^2}\right) - y^2 + \Phi(x) =$

$= \frac{x}{(x^2+y^2)} - y^2 + \Phi(x) \ \Rightarrow$

$\Rightarrow \frac{\partial u}{\partial x} = \frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2} + \Phi'(x) = \frac{y^2-x^2}{(x^2+y^2)^2} + \Phi'(x) \wedge \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow$

$\Rightarrow \frac{y^2-x^2}{(x^2+y^2)^2} + 2x = \frac{y^2-x^2}{(x^2+y^2)^2} + \Phi'(x)$

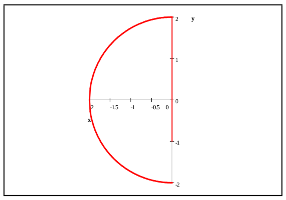
$\Rightarrow \ 2x = \Phi'(x) \Rightarrow \Phi(x) = x^2 + C \Rightarrow u(x,y) = \frac{x}{(x^2+y^2)} + x^2 - y^2 + C$

$f(z) = f(x+iy) = u(x,y) + iv(x,y) = \left(\frac{x}{(x^2+y^2)} + x^2 - y^2 + C\right) + i\left(-\frac{y}{x^2+y^2} + 2xy\right)$

$f(1) = 0 \Rightarrow 0 = f(1+0i) = \left(\frac{1}{(1^2+0^2)} + 1^2 - 0^2 + C\right) + i\left(-\frac{0}{1^2+0^2} + 0\right) =$

$= (2+C) + i(0) \Rightarrow C = -2 \Rightarrow f(z) = \left(\frac{x}{(x^2+y^2)} + x^2 - y^2 - 2\right) + i\left(-\frac{y}{x^2+y^2} + 2xy\right)$

3a. Vypočítajte $\int\limits_C |z| \operatorname{Im} z dz$, kde $C : |z| = 2, \ \operatorname{Re} z \leq 0$ od bodu $-2i$ po bod $2i$ a úsečka od bodu $2i$ po bod $-i$. (**12b**)



$C = C_1 + C_2$

$C_1^- : \varphi_1 : \left\langle \frac{\pi}{2}, \frac{3\pi}{2} \right\rangle \longrightarrow \boldsymbol{C}, \ \varphi_1(t) = 2e^{it}, \ \varphi_1'(t) = 2ie^{it},$

$C_2 : \varphi_2 : \langle 0, 1 \rangle \longrightarrow \boldsymbol{C}, \ \varphi_2(t) = 2i + t(-i-2i) = i(2-3t), \ \varphi_2'(t) = -3i.$

$\int_C |z| \operatorname{Im} z dz = \int_{C_1} |z| \operatorname{Im} z dz + \int_{C_2} |z| \operatorname{Im} z dz =$

$= -\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \underbrace{|2e^{it}|}_{=2} \cdot 2 \underbrace{\sin t}_{\frac{e^{it}-e^{-it}}{2i}} \cdot 2ie^{it} dt + \int_0^1 |i(2-3t)| \cdot (2-3t) \cdot (-3i) \, dt =$

$= -8 \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{e^{it}-e^{-it}}{2i} ie^{it} dt + \int_0^1 |2-3t| \cdot (2-3t) \cdot (-3i) \, dt =$

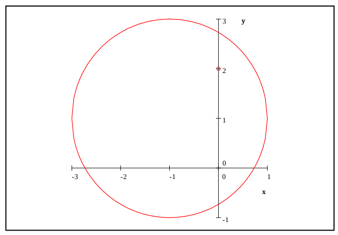
$= \left| |2-3t| = \begin{cases} 2-3t & \text{pre} \quad t \leq \frac{2}{3} \\ -(2-3t) & \text{pre} \quad t > \frac{2}{3} \end{cases} \right| =$

$= -4\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left(e^{2it} - 1\right) dt - 3i\int_0^{\frac{2}{3}} (2-3t)^2\, dt + 3i\int_{\frac{2}{3}}^{1} (2-3t)^2\, dt =$

$= 4\pi - \frac{7}{3}i$

3b. Použitím Cauchyho integrálnej formuly vypočítajte $\int_C \frac{5z^2}{(z-4)(z^2+4)} dz$, $C: |z+1-i| = 2$. **(8b)**

Platí: $2i \in IntC$, $4, -2i \notin IntC$, C je JH krivka, teda platí CIF



$\int_C \frac{5z^2}{(z-4)(z^2+4)} dz = \int_C \frac{5z^2}{(z-4)(z-2i)(z+2i)} dz = \int_C \frac{\frac{5z^2}{(z-4)(z+2i)}}{(z-2i)} dz =$

$= 2\pi i \left[\frac{5z^2}{(z-4)(z+2i)}\right]_{z=2i} = 2\pi i \left(\frac{5(2i)^2}{(4-2i)(4i)}\right) =$

$= 10\pi i \left(\frac{-4}{(4-2i)(4i)}\right) = -5\pi \left(\frac{1}{(2-i)}\right) = -5\pi \left(\frac{2+i}{5}\right) = -(2+i)\,\pi.$

4. Nájdite rozvoj funkcie $f(z) = \frac{1}{z^2-3iz-2}$ do Laurentovho radu v bode $a = i$ na medzikruží $P(2i, 2, \infty)$. **(20b)**

$z^2 - 3iz - 2 = 0 \Rightarrow z_{1,2} = \frac{3i \pm \sqrt{-9+8}}{2} = \frac{3i \pm i}{2} = \begin{cases} 2i \\ i \end{cases}$

$f : \boldsymbol{C} \setminus \{i, 2i\} \longrightarrow \boldsymbol{C}$, $f(z) = \frac{1}{z^2-3iz-2} = \frac{1}{(z-2i)(z-i)} = \frac{i}{z-i} - \frac{i}{z-2i}$

Očak.: $f(z) = \frac{1}{z^2-3iz-2} = \sum_{n=-\infty}^{\infty} c_n (z-2i)^n$.

Funkcia $f(z) = \frac{1}{z^2-3iz-2} = \frac{i}{z-i} - \frac{i}{z-2i}$ je analytická na $P(2i, 1, \infty)$ a platí

$f(z) = \frac{1}{z^2-3iz-2} = \frac{i}{z-i} - \frac{i}{z-2i} = \frac{i}{z+i-2i} - \frac{i}{z-2i} = -\frac{i}{z-2i} + \frac{i}{z-2i}\frac{1}{1+\frac{i}{z-2i}} =$

$= \left[\text{nech } 0 < |z-2i| \wedge \left|-\frac{i}{z-2i}\right| < 1 \Leftrightarrow \underbrace{1 < |z-2i|}_{P(2i,1,\infty)}\right] =$

$= -\frac{i}{z-2i} + \frac{i}{z-2i}\sum_{n=0}^{\infty} (-1)^n \left(\frac{i}{z-2i}\right)^n = -\frac{i}{z-2i} + \sum_{n=0}^{\infty} (-1)^n \left(\frac{i}{z-2i}\right)^{n+1} =$

$= -\frac{i}{z-2i} + \frac{i}{z-2i} + \sum_{n=1}^{\infty} (-1)^n \left(\frac{i}{z-2i}\right)^{n+1} = \sum_{n=1}^{\infty} (-1)^n \left(\frac{i}{z-2i}\right)^{n+1} =$

$= \sum_{k=2}^{\infty} (-1)^{k-1} \left(\frac{i}{z-2i}\right)^k = \sum_{n=-\infty}^{-2} (-1)^{-n-1}\, i^{-n} (z-2i)^n =$

a rad konverguje na $P(2i, 1, \infty)$.

$\frac{1}{(z-2i)(z-i)} = \frac{1}{(z-2i)(z-2i+i)} = \frac{1}{(z-2i)^2\left(1+\frac{i}{z-2i}\right)} = [\text{nech } 0 < |z-2i| \wedge \left|-\frac{i}{z-2i}\right| < 1 \Leftrightarrow \underbrace{1 < |z-2i|}_{P(2i,1,\infty)}] =$

$= \frac{1}{(z-2i)^2\left(1+\frac{i}{z-2i}\right)} = \frac{1}{(z-2i)^2} \sum_{n=0}^{\infty} (-1)^n \left(\frac{i}{z-2i}\right)^n = \sum_{n=0}^{\infty} (-1)^n \, i^n \, (z-2i)^{-n-2}$