1. Vypočítajte integrál priamym integrovaním:

$$\int \frac{1+2x^2}{x^2(1+x^2)}dx$$

Riešenie:

$$\begin{aligned}\int \frac{1+2x^2}{x^2(1+x^2)}dx &= \int \frac{(1+x^2)+x^2}{x^2(1+x^2)}dx = \int \frac{1}{x^2} + \frac{1}{1+x^2}dx = \\ &= \int x^{-2}dx + \int \frac{1}{1+x^2}dx = -x^{-1} + \mathrm{arctg}x + c\end{aligned}$$

2. Vypočítajte integrál priamym integrovaním:

$$\int \frac{1+\cos^2 x}{1+\cos 2x}dx$$

Riešenie:

$$\begin{aligned}\int \frac{1+\cos^2 x}{1+\cos 2x}dx &= \int \frac{1+\cos^2 x}{1+\cos^2 x-\sin^2 x}dx = \int \frac{1+\cos^2 x}{\cos^2 x+\sin^2 x+\cos^2 x-\sin^2 x}dx = \\ &= \int \frac{1+\cos^2 x}{2\cos^2 x}dx = \int \left(\frac{1}{2\cos^2 x} + \frac{1}{2}\right) dx = \frac{1}{2}\int \frac{1}{\cos^2 x}dx + \int \frac{1}{2}\ dx = \\ &= \frac{1}{2}\mathrm{tg}x + \frac{1}{2}x + c\end{aligned}$$

3. Vypočítajte integrál priamym integrovaním:

$$\int \sin 2x \cos 3x dx$$

Riešenie:

Najprv si uvedomme, že:

$$\begin{aligned}\sin(5x) &= \sin(3x+2x) = \sin 3x \cos 2x + \sin 2x \cos 3x \\ \sin x &= \sin(3x-2x) = \sin 3x \cos 2x - \sin 2x \cos 3x\end{aligned}$$

Ich odčítaním dostaneme:

$$\sin(5x) - \sin x = 2\sin 2x \cos 3x$$

Odtiaľ plynie:

$$\sin 2x \cos 3x = \frac{\sin(5x)-\sin x}{2}$$

Potom:

$$\begin{aligned}\int \sin 2x \cos 3x dx &= \int \frac{\sin(5x)-\sin x}{2}dx = \frac{1}{2}\int \sin(5x)dx - \frac{1}{2}\int \sin x\ dx = \\ &= -\frac{1}{2}\frac{\cos(5x)}{5} + \frac{1}{2}\cos x + c = \frac{\cos x}{2} - \frac{\cos 5x}{10} + c\end{aligned}$$

4. Vypočítajte integrál:

$$\int \sin 4x \sin 2x dx$$

Riešenie:

(a) *Priamym integrovaním*:

Odpočítaním nasledujúcich dvoch rovníc

$$\begin{aligned} \cos(2x) &= \cos(4x-2x) = \cos 4x \cos 2x + \sin 4x \sin 2x \\ \cos(6x) &= \cos(4x+2x) = \cos 4x \cos 2x - \sin 4x \sin 2x \end{aligned}$$

dostaneme:

$$\cos(2x) - \cos(6x) = 2 \sin 4x \sin 2x$$

Odtiaľ plynie:

$$\sin 4x \sin 2x = \frac{\cos(2x) - \cos(6x)}{2}$$

Potom:

$$\begin{aligned} \int \sin 4x \sin 2x dx &= \int \frac{\cos(2x) - \cos(6x)}{2} dx = \frac{1}{2} \int \cos(2x) dx - \frac{1}{2} \int \cos(6x) dx = \\ &= \frac{1}{2} \frac{\sin(2x)}{2} - \frac{1}{2} \frac{\sin(6x)}{6} + c = \frac{\sin(2x)}{4} - \frac{\sin(6x)}{12} + c \end{aligned}$$

(b) *Metódou per partes*:

$$I = \int \sin 4x \sin 2x dx = \left| \begin{array}{ll} f(x) = \sin 4x & g'(x) = \sin 2x \\ f'(x) = 4 \cos 4x & g(x) = \frac{-\cos 2x}{2} \end{array} \right| = -\tfrac{\sin 4x \cos 2x}{2} +$$

$$+2 \int \cos 4x \cos 2x dx = \left| \begin{array}{ll} f(x) = \cos 4x & g'(x) = \cos 2x \\ f'(x) = -4 \sin 4x & g(x) = \frac{\sin 2x}{2} \end{array} \right| = -\tfrac{\sin 4x \cos 2x}{2} +$$

$$+ \cos 4x \sin 2x + 4 \int \sin 4x \sin 2x dx = \cos 4x \sin 2x - \tfrac{\sin 4x \cos 2x}{2} + 4I$$

Odtiaľ plynie:

$$\begin{aligned} -3I &= \cos 4x \sin 2x - \frac{\sin 4x \cos 2x}{2} \\ I &= \frac{\sin 4x \cos 2x}{6} - \frac{\cos 4x \sin 2x}{3} + c \\ \int \sin 4x \sin 2x dx &= \frac{\sin 4x \cos 2x}{6} - \frac{\cos 4x \sin 2x}{3} + c \\ & \left(= \frac{1}{6} \frac{\sin 6x + \sin 2x}{2} - \frac{1}{3} \frac{\sin 6x - \sin 2x}{2} + c \right) \\ & \left(= \frac{\sin(2x)}{4} - \frac{\sin(6x)}{12} + c \right) \end{aligned}$$

(c) *Substitučnou metódou*:

$$\int \sin 4x \sin 2x dx = \int 2 \sin 2x \cos 2x \sin 2x dx = \int 2 \sin^2 2x \cos 2x dx =$$

$$= \left| \begin{array}{l} t = \sin 2x \\ dt = 2 \cos 2x \ dx \end{array} \right| = \int t^2 \ dt = \tfrac{t^3}{3} + c = \tfrac{\sin^3(2x)}{3} + c$$

Samozrejme:

$$\begin{aligned} \frac{\sin^3(2x)}{3} &= \frac{\sin 2x \sin^2 2x}{3} = \frac{(\sin 2x)}{3} \frac{(1 - \cos 4x)}{2} = \frac{\sin 2x}{6} - \frac{\sin 2x \cos 4x}{6} = \\ &= \frac{\sin 2x}{6} - \frac{\sin 6x - \sin 2x}{12} = \frac{\sin(2x)}{4} - \frac{\sin(6x)}{12} \end{aligned}$$

5. Vypočítajte integrál:

$$\int \frac{3x^8 + 8x^7 - 9x^6 - 39x^5 + 30x^4 + 68x^3 - 63x^2 + 27x + 35}{x^6 + 3x^5 - 2x^4 - 12x^3 + 3x^2 + 17x - 10} dx$$

Riešenie:

Označme:

$$\begin{aligned} p(x) &:= 3x^8 + 8x^7 - 9x^6 - 39x^5 + 30x^4 + 68x^3 - 63x^2 + 27x + 35 \\ q(x) &:= x^6 + 3x^5 - 2x^4 - 12x^3 + 3x^2 + 17x - 10 \end{aligned}$$

Vydeľme polynómy p a q, dostaneme:

$$\frac{p(x)}{q(x)} = 3x^2 - x + \frac{-5x^5 + 9x^4 + 20x^3 - 16x^2 + 17x + 35}{x^6 + 3x^5 - 2x^4 - 12x^3 + 3x^2 + 17x - 10}$$

Označme ďalej:

$$r(x) := -5x^5 + 9x^4 + 20x^3 - 16x^2 + 17x + 35$$

Potom:

$$\int \frac{p(x)}{q(x)} dx = \int 3x^2 - x + \frac{r(x)}{q(x)} dx = \int 3x^2 - x\ dx + \int \frac{r(x)}{q(x)} dx$$

Riešme integrály zvlášť. Najprv:

$$\int 3x^2 - x\ dx = x^3 - \frac{x^2}{2} + k \tag{1}$$

Pri riešení druhého integrálu treba najprv rozložiť polynóm v menovateli, teda $q(x)$, nad $\mathbb{R}$. Môžeme na to použiť Hornerovu schému:

	1	3	-2	-12	3	17	-10
1		1	4	2	-10	-7	10
	1	4	2	-10	-7	10	
1		1	5	7	-3	-10	
	1	5	7	-3	-10		
1		1	6	13	10		
	1	6	13	10			
-2		-2	-8	-10			
	1	4	5				

Teda máme:

$$q(x) = (x-1)^3(x+2)(x^2 + 4x + 5) \tag{2}$$

S využitím (2) urobme rozklad na súčet elementárnych zlomkov:

$$\frac{r(x)}{q(x)} = \frac{A}{(x-1)^3} + \frac{B}{(x-1)^2} + \frac{C}{x-1} + \frac{D}{x+2} + \frac{Ex + F}{x^2 + 4x + 5}$$

Hodnoty koeficientov A, B, C, D, E, F nájdeme z rovnosti:

$$\begin{aligned} r(x) = {} & A(x+2)(x^2+4x+5) + B(x-1)(x+2)(x^2+4x+5) + \\ & + C(x-1)^2(x+2)(x^2+4x+5) + D(x-1)^3(x^2+4x+5) + \\ & + (Ex+F)(x-1)^3(x+2) \end{aligned}$$

Porovnaním polynómov na oboch stranách rovnice, získame:

$$\begin{aligned}
-5 &= C + D + E \\
9 &= B + 4C + D - E + F \\
20 &= A + 5B + 2C - 4D - 3E - F \\
-16 &= 6A + 7B - 10C - 4D + 5E - 3F \\
17 &= 13A - 3B - 7C + 11D - 2E + 5F \\
35 &= 10A - 10B + 10C - 5D - 2F
\end{aligned}$$

Odtiaľ:

$$A = 2,\ B = 0,\ C = 1,\ D = -3,\ E = -3,\ F = 5.$$

Riešme druhý z integrálov:

$$\begin{aligned}
\int \frac{r(x)}{q(x)}dx &= \int \frac{2}{(x-1)^3} + \frac{1}{x-1} - \frac{3}{x+2} + \frac{-3x+5}{x^2+4x+5}dx \\
&= -\frac{1}{(x-1)^2} + \ln|x-1| - 3\ln|x+2| - 3\int \frac{x-\frac{5}{3}}{x^2+4x+5}dx \\
&= -\frac{1}{(x-1)^2} + \ln|x-1| - 3\ln|x+2| - \frac{3}{2}\int \frac{2x-\frac{10}{3}}{x^2+4x+5}dx \\
&= -\frac{1}{(x-1)^2} + \ln|x-1| - 3\ln|x+2| - \frac{3}{2}\int \frac{2x+4-\frac{22}{3}}{x^2+4x+5}dx \\
&= -\frac{1}{(x-1)^2} + \ln|x-1| - 3\ln|x+2| - \frac{3}{2}\int \frac{2x+4}{x^2+4x+5}dx + 11\int \frac{1}{x^2+4x+5}dx \\
&= -\frac{1}{(x-1)^2} + \ln|x-1| - 3\ln|x+2| - \frac{3}{2}\ln(x^2+4x+5) + 11\int \frac{1}{(x+2)^2+1^2}dx \\
&= -\frac{1}{(x-1)^2} + \ln\left|\frac{x-1}{(x+2)^3}\right| - \frac{3}{2}\ln(x^2+4x+5) + 11\mathrm{arctg}(x+2) + K \qquad (3)
\end{aligned}$$

Súčtom výsledkov (1) a (3) dostaneme:

$$\int \frac{3x^8+8x^7-9x^6-39x^5+30x^4+68x^3-63x^2+27x+35}{x^6+3x^5-2x^4-12x^3+3x^2+17x-10}dx = x^3 - \frac{x^2}{2} +$$
$$+\frac{-1}{(x-1)^2} + \ln\left|\frac{x-1}{(x+2)^3}\right| - \frac{3}{2}\ln(x^2+4x+5) + 11\mathrm{arctg}(x+2) + c$$

6. Vypočítajte integrál:

$$\int \frac{1}{x^4+4}dx$$

Riešenie:

$$\int \frac{1}{x^4+4}dx = \int \frac{1}{x^4+4x^2+4-4x^2}dx = \int \frac{1}{(x^2+2)^2-(2x)^2}dx =$$
$$= \int \frac{1}{(x^2+2x+2)(x^2-2x+2)}dx = \int \frac{Ax+B}{x^2+2x+2} + \frac{Cx+D}{x^2-2x+2}dx \qquad (4)$$

Hodnoty koeficientov A, B, C, D nájdeme z rovnosti:

$$1 = (Ax+B)(x^2-2x+2) + (Cx+D)(x^2+2x+2)$$

Porovnaním polynómov na oboch stranách rovnice, získame:

$$\begin{aligned}
0 &= A + C\\
0 &= -2A + B + 2C + D\\
0 &= 2A - 2B + 2C + 2D\\
1 &= 2B + 2D
\end{aligned}$$

Riešením sústavy je:

$$A = \frac{1}{8},\ B = \frac{1}{4},\ C = -\frac{1}{8},\ D = \frac{1}{4}.$$

Po dosadení do (4) dostaneme:

$$\begin{aligned}
\int \frac{1}{x^4+4}dx &= \frac{1}{8}\int \frac{x+2}{x^2+2x+2} - \frac{x-2}{x^2-2x+2}dx = \frac{1}{16}\int \frac{2x+4}{x^2+2x+2} - \frac{2x-4}{x^2-2x+2}dx =\\
&= \frac{1}{16}\int \frac{2x+2+2}{x^2+2x+2} - \frac{2x-2-2}{x^2-2x+2}dx = \frac{1}{16}\int \frac{2x+2}{x^2+2x+2}dx\\
&\quad + \frac{1}{8}\int \frac{1}{x^2+2x+2}dx - \frac{1}{16}\int \frac{2x-2}{x^2-2x+2}dx + \frac{1}{8}\int \frac{1}{x^2-2x+2}dx =\\
&= \frac{1}{16}\ln(x^2+2x+2) + \frac{1}{8}\int \frac{1}{(x+1)^2+1}dx -\\
&\quad - \frac{1}{16}\ln(x^2-2x+2) + \frac{1}{8}\int \frac{1}{(x-1)^2+1}dx =\\
&= \frac{1}{16}\ln\left(\frac{x^2+2x+2}{x^2-2x+2}\right) + \frac{1}{8}\operatorname{arctg}(x+1) + \frac{1}{8}\operatorname{arctg}(x-1) + c
\end{aligned}$$

7. Vypočítajte integrál:

$$\int \frac{x}{x^8-1}dx$$

Riešenie:

$$\int \tfrac{x}{x^8-1}dx = \tfrac{1}{2}\int \tfrac{2x}{x^8-1}dx = \left| \begin{array}{l} t = x^2 \\ dt = 2x\ dx \end{array} \right| = \tfrac{1}{2}\int \tfrac{1}{t^4-1}dt = \tfrac{1}{2}\int \tfrac{1}{(t-1)(t+1)(t^2+1)}dt$$

Upravme integrand na súčet elementárnych zlomkov nad $\mathbb{R}$:

$$\frac{1}{2}\int \frac{1}{(t-1)(t+1)(t^2+1)}dt = \frac{1}{2}\int \frac{A}{t-1} + \frac{B}{t+1} + \frac{Ct+D}{t^2+1}dt \qquad (5)$$

Hodnoty koeficientov A, B, C, D nájdeme z rovnosti:

$$1 = A(t+1)(t^2+1) + B(t-1)(t^2+1) + (Ct+D)(t^2-1)$$

Porovnaním polynómov na oboch stranách rovnice, získame:

$$\begin{aligned}
0 &= A + B + C\\
0 &= A - B + D\\
0 &= A + B - C\\
1 &= A - B - D
\end{aligned}$$

Riešením sústavy je:

$$A = \frac{1}{4},\ B = -\frac{1}{4},\ C = 0,\ D = -\frac{1}{2}.$$

Po dosadení do (5) dostaneme:

$$\begin{aligned}\int \frac{x}{x^8-1}dx &= \frac{1}{8}\int \frac{1}{t-1}dt - \frac{1}{8}\int \frac{1}{t+1}dt - \frac{1}{4}\int \frac{1}{t^2+1}dt \\ &= \frac{1}{8}\ln|t-1| - \frac{1}{8}\ln|t+1| - \frac{1}{4}\text{arctg } t + c = \\ &= \frac{1}{8}\ln|x^2-1| - \frac{1}{8}\ln|x^2+1| - \frac{1}{4}\text{arctg } x^2 + c\end{aligned}$$

8. Vypočítajte

$$\int_{-1}^{1} \frac{1}{(\mathrm{e}^x+1)(x^2+1)}dx.$$

Riešenie:

$$\begin{aligned}\int_{-1}^{1} \frac{1}{(\mathrm{e}^x+1)(x^2+1)}dx &= \left| \begin{array}{ll} f(x) = \frac{1}{\mathrm{e}^x+1} & g'(x) = \frac{1}{x^2+1} \\ f'(x) = \frac{-\mathrm{e}^x}{(\mathrm{e}^x+1)^2} & g(x) = \text{arctg}x \end{array} \right| = \left[\frac{\text{arctg}x}{\mathrm{e}^x+1}\right]_{-1}^{1} + \\ &+ \int_{-1}^{1} \frac{\mathrm{e}^x\text{arctg}x}{(\mathrm{e}^x+1)^2}dx = \left[\frac{\mathrm{e}^x\text{arctg}x}{(\mathrm{e}^x+1)^2} \text{ je nepárna funkcia } (\textit{overte})\right] = \\ &= \frac{\pi}{4(\mathrm{e}^x+1)} - \frac{-\pi\mathrm{e}^x}{4(\mathrm{e}^x+1)} + 0 = \frac{\pi}{4}\end{aligned}$$

9. Vypočítajte obsah plochy ohraničenej parabolou $y = x^2 - 6x + 8$ a jej dotyčnicami v bodoch $A = [1, 3]$, $B = [4, 0]$.

Riešenie:

Vypočítajme najprv obe dotyčnice. Označme $f(x) = x^2 - 6x + 8$, potom $f'(x) = 2x - 6$ a $f'(1) = -4$, $f'(4) = 2$. Potom rovnice dotyčníc sú:

$$t_A: \quad y - 3 = -4(x-1) \qquad t_B: \quad y - 0 = 2(x-4)$$
$$y = -4x + 7 \qquad\qquad y = 2x - 8$$

a ich priesečník je bod $C = \left[\frac{5}{2}, -3\right]$. Obsah plochy teda vypočítame ako súčet integrálov:

$$\int_1^{\frac{5}{2}} x^2 - 6x + 8 - (-4x+7)\ dx + \int_{\frac{5}{2}}^{4} x^2 - 6x + 8 - (2x-8)\ dx = \int_1^{\frac{5}{2}} x^2 - 2x + 1\ dx +$$
$$+ \int_{\frac{5}{2}}^{4} x^2 - 8x + 16\ dx = \int_1^{\frac{5}{2}} (x-1)^2\ dx + \int_{\frac{5}{2}}^{4} (x-4)^2\ dx = \left[\frac{(x-1)^3}{3}\right]_1^{\frac{5}{2}} + \left[\frac{(x-4)^3}{3}\right]_{\frac{5}{2}}^{4} =$$
$$= \frac{1}{3}\left(\left(\frac{3}{2}\right)^3 - 0 + 0 - \left(-\frac{3}{2}\right)^3\right) = \frac{1}{3}\left(\left(\frac{3}{2}\right)^3 + \left(\frac{3}{2}\right)^3\right) = \frac{2}{3}\left(\frac{3}{2}\right)^3 = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$$

10. Vypočítajte objem telesa, ktoré vznikne rotáciou elementárnej oblasti D okolo osi $\mathcal{O}_y$, kde D je určená krivkami $y = x$, $y = x + \sin^2 x$, $x = 0$, $x = \pi$.

Riešenie:

Objem tohto telesa dostaneme vypočítaním integrálu:

$$2\pi \int_0^\pi x(x+\sin^2 x - x)dx = 2\pi \int_0^\pi x\sin^2 x\ dx = \pi \int_0^\pi x(1-\cos 2x)dx =$$
$$= \pi \int_0^\pi x\ dx - \pi \int_0^\pi \cos 2x\ dx = \pi \left[\frac{x^2}{2}\right]_0^\pi - \pi \left[\frac{\sin 2x}{2}\right]_0^\pi = \frac{\pi^3}{2}$$

11. Vypočítajte dĺžku krivky $C = \{(x,y) \in \mathbb{R}\times\mathbb{R}; x \in \langle a,b\rangle, y = f(x)\}$, ak $a = 0$, $b = 1$ a $f(x) = \sqrt{e^{2x}-1} - \mathrm{arctg}\sqrt{e^{2x}-1}$.

Riešenie:

Vypočítajme najprv deriváciu funkcie f:

$$f'(x) = \frac{e^{2x}}{\sqrt{e^{2x}-1}} - \frac{e^{2x}}{\sqrt{e^{2x}-1}}\frac{1}{(1+e^{2x}-1)} = \frac{e^{2x}-1}{\sqrt{e^{2x}-1}} = \sqrt{e^{2x}-1}$$

Dosaďme do vzťahu pre výpočet dĺžky krivky:

$$\int_0^1 \sqrt{1+(\sqrt{e^{2x}-1})^2}\ dx = \int_0^1 \sqrt{1+e^{2x}-1}\ dx = \int_0^1 e^x dx = [e^x]_0^1 = e-1$$

12. Vypočítajte dĺžku krivky $C = \{(x,y) \in \mathbb{R}\times\mathbb{R}; y \in \langle a,b\rangle, x = g(y)\}$, ak $y \in \langle 1, e\rangle$ a $g(y) = \frac{1}{4}y^2 - \frac{1}{2}\ln y$.

Riešenie:

Vypočítajme najprv deriváciu funkcie g:

$$g'(y) = \frac{y}{2} - \frac{1}{2y}$$

Dosaďme do vzťahu pre výpočet dĺžky krivky:

$$\int_1^e \sqrt{1+\left(\frac{y}{2}-\frac{1}{2y}\right)^2}\ dy = \int_1^e \sqrt{1+\frac{1}{4}\left(y^2-2+\frac{1}{y^2}\right)}\ dy = \int_1^e \sqrt{\frac{1}{4}\left(y^2+2+\frac{1}{y^2}\right)}\ dy =$$
$$= \frac{1}{2}\int_1^e \sqrt{\left(y+\frac{1}{y}\right)^2}\ dy = \frac{1}{2}\int_1^e \left(y+\frac{1}{y}\right)\ dy = \left[\frac{y^2}{4}\right]_1^e + \left[\frac{\ln y}{2}\right]_1^e = \frac{e^2-1}{4} + \frac{1}{2} = \frac{e^2+1}{4}$$