Príklady z M2E

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1 Druhý týždeň

V cvičeniach 1 - 19 vypočítajte integrály priamym integrovaním (pomocou integrálov elementárnych funkcií):

1. $\int \left(3x^3 + 2x - 4\right) dx.$ $\left[\frac{3}{4}x^4 + x^2 - 4x + C\right]$
2. $\int \left(\frac{1}{3}x^2 - \frac{1}{5}x\right) dx.$ $\left[\frac{1}{9}x^3 - \frac{1}{10}x^2 + C\right]$
3. $\int \left(\sqrt{x^3} - \frac{1}{\sqrt{x}}\right) dx.$ $\left[\frac{2}{5}x^{\frac{5}{2}} - 2x^{\frac{1}{2}} + C\right]$
4. $\int \frac{\sqrt{x^4+2+x^{-4}}}{x^3} dx.$ $\left[\ln|x| - \frac{1}{4x^4} + C\right]$
5. $\int \frac{x(\sqrt[3]{x} - x\sqrt[3]{x})}{\sqrt[4]{x}} dx.$ $\left[-\frac{12}{37}\sqrt[12]{x^{37}} + \frac{12}{25}\sqrt[12]{x^{25}} + C\right]$
6. $\int \frac{x^3-1}{x-1} dx.$ $\left[\frac{1}{3}x^3 + \frac{1}{2}x^2 + x + C\right]$
7. $\int e^x a^x dx.$ $\left[\frac{e^x a^x}{1+\ln a} + C\right]$
8. $\int \left(5\cos x - 2x^5 + \frac{3}{1+x^2}\right) dx.$ $\left[5\sin x - \frac{1}{3}x^6 + 3\operatorname{arctg} x + C\right]$
9. $\int \left(10^{-x} + \frac{x^2+2}{x^2+1}\right) dx.$ $\left[-\frac{10^{-x}}{\ln 10} + x + \operatorname{arctg} x + C\right]$
10. $\int (2\sin x - 3\cos x)\, dx.$ $[-2\cos x - 3\sin x + C]$
11. $\int \frac{1}{\sqrt{3-3x^2}} dx.$ $\left[\frac{1}{\sqrt{3}} \arcsin x + C\right]$
12. $\int \frac{3.2^x - 2.3^x}{2^x} dx.$ $\left[3x - \frac{3^x}{2^{x-1}(\ln 3 - \ln 2)} + C\right]$
13. $\int \frac{1+\cos^2 x}{1+\cos 2x} dx.$ $\left[\frac{1}{2}\operatorname{tg} x + \frac{1}{2}x + C\right]$
14. $\int \frac{\cos 2x}{\cos^2 x \sin^2 x} dx.$ $[-\operatorname{cotg} x - \operatorname{tg} x + C.]$
15. $\int \operatorname{tg}^2 x dx.$ $[\operatorname{tg} x - x + C]$
16. $\int \operatorname{cotg}^2 x dx.$ $[-\operatorname{cotg} x - x + C]$
17. $\int \frac{dx}{\cos 2x + \sin^2 x}.$ $[\operatorname{tg} x + C]$
18. $\int \frac{(1+2x^2)}{x^2(1+x^2)} dx.$ $\left[-\frac{1}{x} + \operatorname{arctg} x + C\right]$
19. $\int \frac{(1+x)^2}{x(1+x^2)} dx.$ $[\ln|x| + 2\operatorname{arctg} x + C]$
20. $\int 2\sin^2\left(\frac{x}{2}\right) dx.$ $[x - \sin x + C]$
21. $\int_0^1 \frac{1}{1+x^2} dx.$ $\left[\frac{\pi}{4}\right]$
22. $\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1}{\sqrt{1-x^2}} dx.$ $\left[\frac{\pi}{6}\right]$

2 Tretí týždeň

Vypočítajte integrály:

1. $\int x \operatorname{arctg} x dx.$ $\left[\begin{array}{c}\text{Metóda: per partes} \\ f(x)=\operatorname{arctg} x, f'(x)=\frac{1}{1+x^2}, g'(x)=x, g(x)=\frac{x^2}{2} \\ \text{Výsledok :} \frac{x^2}{2}\operatorname{arctg} x+\frac{1}{2}\operatorname{arctg} x-\frac{1}{2}x+C\end{array}\right]$

2. $\int x e^{2x} dx.$ $\left[\begin{array}{c}\text{Metóda: per partes 2x} \\ f(x)=x, f'(x)=1, g'(x)=e^{2x}, g(x)=\frac{1}{2}e^{2x} \\ \text{Výsledok : } \frac{1}{2}xe^{2x}-\frac{1}{4}e^{2x}+C\end{array}\right]$

3. $\int\limits_0^{\frac{\pi}{2}} e^{2x}\sin x dx.$ $\left[\begin{array}{c}\text{Metóda: per partes 2x} \\ f(x)=e^{2x}, f'(x)=2e^{2x}, g'(x)=\sin x, g(x)=-\cos x \\ \text{Výsledok : } \frac{1}{5}\left(1+2e^{\pi}\right)\end{array}\right]$

4. $\int \ln x dx.$ $\left[\begin{array}{c}\text{Metóda: per partes} \\ f(x)=\ln x, f'(x)=\frac{1}{x}, g'(x)=1, g(x)=x \\ \text{Výsledok : } x\ln x-x+C\end{array}\right]$

5. $\int_1^{\sqrt{3}} x \operatorname{arctg} x dx.$ $\left[\begin{array}{c}\text{Metóda: per partes} \\ f(x)=\operatorname{arctg} x, f'(x)=\frac{1}{1+x^2}, g'(x)=x, g(x)=\frac{x^2}{2} \\ \text{Výsledok : } \frac{5\pi}{12}-\frac{1}{2}\left(\sqrt{3}-1\right)\end{array}\right]$

6. $\int \operatorname{arccotg} x dx.$ $\left[\begin{array}{c}\text{Metóda: per partes} \\ f(x)=\operatorname{arccotg} x, f'(x)=-\frac{1}{1+x^2}, g'(x)=1, g(x)=x \\ \text{Výsledok : } x\operatorname{arccotg} x+\frac{1}{2}\ln\left(x^2+1\right)+C\end{array}\right]$

7. $\int \ln^2 x dx.$ $\left[\begin{array}{c}\text{Metóda: per partes 2 krát} \\ f(x)=\ln^2 x, f'(x)=2\ln x\cdot\frac{1}{x}, g'(x)=1, g(x)=x \\ \text{Výsledok : } x\ln^2 x-2x\ln x+2x+C\end{array}\right]$

8. $\int \frac{x}{\cos^2 x} dx.$ $\left[\begin{array}{c}\text{Metóda: per partes} \\ f(x)=x, f'(x)=1, g'(x)=\frac{1}{\cos^2 x}, g(x)=\operatorname{tg} x \\ \text{Výsledok : } x\operatorname{tg} x+\ln|\cos x|+C\end{array}\right]$

9. $\int\limits_0^{\frac{\sqrt{2}}{2}} \frac{\arcsin x}{\sqrt{1+x}} dx.$ $\left[\begin{array}{c}\text{Metóda: per partes} \\ f(x)=\arcsin x, f'(x)=\frac{1}{\sqrt{1-x^2}}, g'(x)=(1+x)^{-\frac{1}{2}}, g(x)=2\sqrt{1+x} \\ \text{Výsledok : } \frac{\pi}{2}\sqrt{1+\frac{\sqrt{2}}{2}}+4\sqrt{1-\frac{\sqrt{2}}{2}}-4\end{array}\right]$

10. $\int x\operatorname{tg}^2 x dx$ $\left[\begin{array}{c}\int x\operatorname{tg}^2 x dx=\int x\left(\frac{1}{\cos^2 x}-1\right)dx+\text{Metóda: per partes} \\ f(x)=x, f'(x)=1, g'(x)=\frac{1}{\cos^2 x}-1, g(x)=\operatorname{tg} x-x \\ \text{Výsledok : } -\frac{1}{2}x^2+x\operatorname{tg} x+\ln|\cos x|+C\end{array}\right]$

11. $\int \cos(\ln x) dx$ $\left[\begin{array}{c}\text{Metóda: per partes 2 krát} \\ f(x)=\cos(\ln x), f'(x)=-\sin(\ln x)\frac{1}{x}, g'(x)=1, g(x)=x \\ \text{Výsledok : } \frac{x}{2}\left(\cos(\ln x)+\sin(\ln x)\right)+C\end{array}\right]$

12. $\int_0^{\pi} x^2 \sin x dx$ $\left[\begin{array}{c}\text{Metóda: per partes 2 krát} \\ f(x)=x^2, f'(x)=2x, g'(x)=\sin x, g(x)=-\cos x \\ \text{Výsledok}: \pi^2-4\end{array}\right]$

13. $\int \operatorname{arctg} \frac{1}{x-1} dx$ $\left[\begin{array}{c}\text{Metóda: per partes + integrovanie racionálnych funkcií} \\ f(x)=\operatorname{arctg} \frac{1}{x-1}, f'(x)=\frac{1}{x^2-2x+2}, g'(x)=1, g(x)=x \\ \text{Výsledok}: x \operatorname{arctg} \frac{1}{x-1}+\frac{1}{2} \ln \left|x^2-2x+2\right|+\operatorname{arctg}(x-1)+C\end{array}\right]$

14. $\int x^2 \arcsin x dx$ $\left[\begin{array}{c}\text{Metóda: per partes} \\ f(x)=\arcsin x, f'(x)=\frac{1}{\sqrt{1-x^2}}, g'(x)=x^2, g(x)=\frac{x^3}{3} \\ \text{Výsledok}: \frac{1}{3} x^3 \arcsin x+\frac{1}{9} x^2 \sqrt{(1-x^2)}+\frac{2}{9} \sqrt{(1-x^2)}+C\end{array}\right]$

15. $\int_0^{\frac{\pi}{4}} \exp(3x) \sin(4x) dx$ $\left[\begin{array}{c}\text{Metóda: per partes 2 krát} \\ f(x)=e^{3x}, f'(x)=3e^{3x}, g'(x)=\sin(4x), g(x)=-\frac{1}{4} \cos(4x) \\ \text{Výsledok}: \frac{4}{25}\left(e^{\frac{3}{4}\pi}+1\right)\end{array}\right]$

16. $\int_0^{\frac{\sqrt{2}}{2}} \frac{\arcsin x}{\sqrt{1+x}} dx$ $\left[\begin{array}{c}\text{Metóda: per partes} \\ f(x)=\arcsin x, f'(x)=\frac{1}{\sqrt{1-x^2}}, g'(x)=(1+x)^{-\frac{1}{2}}, g(x)=2(1+x)^{\frac{1}{2}} \\ \text{Výsledok}: \frac{\pi}{2} \sqrt{\frac{2+\sqrt{2}}{2}}+4 \sqrt{\frac{2-\sqrt{2}}{2}}-4\end{array}\right]$

17. $\int \left(x^2+x\right) \ln(x+1) dx$ $\left[\begin{array}{c}\text{Metóda: substitučná + per partes+integrovanie racionálnych funkcií} \\ t=x+1, dt=dx \\ \text{Výsledok}: \frac{2(x+1)^3-3(x+1)^2}{6} \ln(x+1)-\frac{4(x+1)^3-9(x+1)^2}{36}+C\end{array}\right]$

18. $\int \arcsin\left(\sqrt{\frac{x}{x+1}}\right) dx$ $\left[\begin{array}{c}\text{Metóda: per partes} \\ f(x)=\arcsin\left(\sqrt{\frac{x}{x+1}}\right), f'(x)=\frac{1}{2\sqrt{x}(x+1)}, g'(x)=1, g(x)=x \\ \text{Výsledok}: x \arcsin \sqrt{\frac{x}{x+1}}-\sqrt{x}+\operatorname{arctg} \sqrt{x}+C\end{array}\right]$

3 Štvrtý týždeň

Vypočítajte integrály

1. $\int \frac{-2x+19}{x^2+x-6}dx.$ $\left[\begin{array}{c} \text{Metóda: integrovanie racionálnych funkcií.} \\ \frac{-2x+19}{x^2+x-6} = \frac{3}{x-2} - \frac{5}{x+3} \\ \text{Výsledok: } \ln\frac{|x-2|^3}{|x+3|^5} + C \end{array}\right]$

2. $\int \frac{2}{9x^2-1}dx.$ $\left[\begin{array}{c} \text{Metóda: integrovanie racionálnych funkcií.} \\ \frac{2}{(3x+1)(3x-1)} = \frac{1}{3x-1} - \frac{1}{3x+1} = \frac{1}{3}\frac{1}{x-\frac{1}{3}} - \frac{1}{3}\frac{1}{x+\frac{1}{3}} \\ \text{Výsledok: } \frac{1}{3}\ln\left(x-\frac{1}{3}\right) - \frac{1}{3}\ln\left(x+\frac{1}{3}\right) + C \end{array}\right]$

3. $\int \frac{5x^2-7x+10}{x^3-x^2-4x-6}dx.$ $\left[\begin{array}{c} \text{Metóda: integrovanie racionálnych funkcií.} \\ \frac{5x^2-7x+10}{x^3-x^2-4x-6} = \frac{2}{x-3} + \frac{3x-2}{x^2+2x+2} = \\ = \frac{2}{x-3} + \frac{3}{2}\frac{2x-\frac{4}{3}}{x^2+2x+2} = \frac{2}{x-3} + \frac{3}{2}\frac{2x+2-2-\frac{4}{3}}{x^2+2x+2} = \\ = \frac{2}{x-3} + \frac{3}{2}\frac{2x+2-2-\frac{4}{3}}{x^2+2x+2} = \frac{2}{x-3} + \frac{3}{2}\frac{2x+2}{x^2+2x+2} - 5\frac{1}{(x+1)^2+1} \\ \text{Výsledok: } 2\ln|x-3| + \frac{3}{2}\ln\left(x^2+2x+2\right) - 5\,\text{arctg}\,(x+1) + C \end{array}\right]$

4. $\int \frac{4x^2+x-13}{2x^3+12x^2+11x+5}dx.$ $\left[\begin{array}{c} \text{Metóda: integrovanie racionálnych funkcií.} \\ \frac{4x^2+x-13}{2x^3+12x^2+11x+5} = \frac{4x^2+x-13}{(x+5)(2x^2+2x+1)} = \frac{2}{x+5} - \frac{3}{2x^2+2x+1} = \\ = \frac{2}{x+5} - \frac{3}{2}\frac{1}{x^2+x+\frac{1}{2}} = \frac{2}{x+5} - \frac{3}{2}\frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{1}{4}} = \frac{2}{x+5} - \frac{\frac{3}{2}}{\frac{1}{4}}\frac{1}{\frac{\left(x+\frac{1}{2}\right)^2}{\left(\frac{1}{2}\right)^2}+1} = \\ = \frac{2}{x+5} - 6\frac{1}{(2x+1)^2+1} \\ \text{Výsledok: } 2\ln|x+5| - 3\,\text{arctg}\,(2x+1) + C \end{array}\right]$

5. $\int \frac{2x+1}{x^2+2x+5}dx.$ $\left[\begin{array}{c} \text{Metóda: integrovanie racionálnych funkcií.} \\ \frac{2x+1}{x^2+2x+5} = \frac{2x+2}{x^2+2x+5} - \frac{1}{(x+1)^2+4} \\ \text{Výsledok: } \ln|x^2+2x+5| - \frac{1}{2}\,\text{arctg}\,\frac{x+1}{2} + C \end{array}\right]$

6. $\int \frac{1}{x^3+1}dx$ $\left[\begin{array}{c} \text{Metóda: integrovanie racionálnych funkcií.} \\ \frac{1}{x^3+1} = \frac{1}{(x+1)(x^2-x+1)} = \frac{1}{3(x+1)} + \frac{1}{x^2-x+1}\left(\frac{2}{3} - \frac{1}{3}x\right) \\ \text{Výsledok: } \frac{1}{3}\ln|x+1| - \frac{1}{6}\ln\left(x^2-x+1\right) + \frac{\sqrt{3}}{3}\,\text{arctg}\,\frac{\sqrt{3}}{3}(2x-1) + C. \end{array}\right]$

7. $\int \frac{x^5+x^4-7x^3+8x-3}{x^3+x^2-6x}dx.$ $\left[\begin{array}{c} \text{Metóda: integrovanie racionálnych funkcií.} \\ \frac{x^5+x^4-7x^3+8x-3}{x^3+x^2-6x} = x^2 - 1 + \frac{1}{2x} + \frac{1}{2(x-2)} \\ \text{Výsledok: } \frac{x^3}{3} - x + \frac{1}{2}\ln|x(x-2)| + C \end{array}\right]$

8. $\int \frac{6x-13}{(4x^2+4x+17)^2}dx.$ $\left[\begin{array}{c} \text{Metóda: integrovanie racionálnych funkcií.} \\ \text{Výsledok:} -\frac{x+2}{2(4x^2+4x+17)} - \frac{1}{16}\,\text{arctg}\,\frac{2x+1}{4} + C \end{array}\right]$

9. $\int \frac{1}{x^3+1}dx$ $\left[\begin{array}{c} \text{Metóda: integrovanie racionálnych funkcií.} \\ \text{Výsledok: } \frac{1}{3}\ln|x+1| - \frac{1}{6}\ln\left(x^2-x+1\right) + \frac{\sqrt{3}}{3}\,\text{arctg}\,\frac{\sqrt{3}}{3}(2x-1) + C. \end{array}\right]$

10. $\int \frac{x^5+x^4-7x^3+8x-3}{x^3+x^2-6x}dx.$ $\left[\begin{array}{c} \text{Metóda: integrovanie racionálnych funkcií.} \\ \text{Výsledok: } \frac{x^3}{3} - x + \frac{1}{2}\ln|x(x-2)| + C \end{array}\right]$

11. $\int \frac{2x^3-2x^2+4x-4}{x^4+4}dx.$ $\left[\begin{array}{c} \text{Návod: } x^4+4 = x^4+4x^2+4-4x^2 = (x^2+2)^2-(2x)^2. \\ \text{Metóda:integrovanie racionálej funkcie.} \\ \text{Výsledok:}\frac{1}{2}\ln\left[\left(x^2-2x+2\right)\left(x^2+2x+2\right)\right] - 2\operatorname{arctg}(x+1) + C \end{array}\right]$

12. $\int \frac{7-x}{x^3-x^2+3x+5}dx$ $\left[\begin{array}{c} \text{Metóda: integrovanie racionálnych funkcií.} \\ \text{Výsledok:} \ln\frac{|x+1|}{\sqrt{x^2-2x+5}} + \frac{1}{2}\operatorname{arctg}\left(\frac{x-1}{2}\right) + C. \end{array}\right]$

13. $\int \frac{\ln\left(1-x+x^2\right)}{x^2}dx$ $\left[\begin{array}{c} \text{Metóda: per partes+integrovanie racionálnych funkcií} \\ f(x) = \ln\left(1-x+x^2\right), f'(x) = \frac{2x-1}{1-x+x^2}, g'(x) = x^{-2}, g(x) = \frac{x^{-1}}{(-1)} \\ \text{Výsledok: } -\frac{1}{x}\ln\left(x^2-x+1\right) - \ln|x| + \frac{1}{2}\ln\left(x^2-x+1\right) + \sqrt{3}\operatorname{arctg}\frac{\sqrt{3}}{3}(2x-1) + C \end{array}\right]$

14. $\int \sin x \ln(\operatorname{tg} x)\,dx$ $\left[\begin{array}{c} \text{Metóda: per partes+substitúcia} \\ f(x) = \ln(\operatorname{tg} x), f'(x) = \frac{1}{\sin x\cos x}, g'(x) = \sin x, g(x) = -\cos x \\ \text{Výsledok: } -\cos x \ln(\operatorname{tg} x) + \ln\left|\operatorname{tg}\frac{x}{2}\right| + C \end{array}\right]$

15. $\int \arccos x dx.$ $\left[\begin{array}{c} \text{Metóda: per partes + substitúcia} \\ f(x) = \arccos x, f'(x) = -\frac{1}{\sqrt{1-x^2}}, g'(x) = 1, g(x) = x \\ \text{Výsledok: } x\arccos x - \sqrt{1-x^2} + C \end{array}\right]$

16. $\int \frac{\cos x}{\sin^2 x+5\sin x-6}dx.$ $\left[\begin{array}{c} \text{Metóda: substitučná + integrovanie racionálnych funkcií} \\ t = \sin x, dt = \cos x dx \\ \text{Výsledok: } \frac{1}{7}\ln\frac{1-\sin x}{\sin x+6} \end{array}\right]$

17. $\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin^2 x+\sin x-6}dx.$ $\left[\begin{array}{c} \text{Metóda: substitučná + integrovanie racionálnych funkcií} \\ t = \sin x, dt = \cos x dx, x = 0 \to\ t = 0, x = \frac{\pi}{2} \to t = 1 \\ \text{Výsledok: } \frac{1}{5}\ln\frac{3}{8} \end{array}\right]$

18. $\int_0^{\frac{\pi}{2}} \frac{\sin x}{6-5\cos x+\cos^2 x}dx.$ $\left[\begin{array}{c} \text{Metóda: substitučná + integrovanie racionálnych funkcií} \\ t = \cos x, dt = -\sin x dx, x = 0 \to\ t = 1, x = \frac{\pi}{2} \to t = 0 \\ \text{Výsledok: } \ln\frac{4}{3} \end{array}\right]$

19. $\int \frac{e^x(e^x-1)}{e^{2x}+1}dx.$ $\left[\begin{array}{c} \text{Metóda: substitučná + integrovanie racionálnych funkcií} \\ t = e^x, dt = e^x dx \\ \text{Výsledok: } \frac{1}{2}\ln\left(e^{2x}+1\right) - \operatorname{arctg}(e^x) + C \end{array}\right]$

20. $\int \frac{-2\sin x}{\cos^2 x+2\cos x+5}dx.$ $\left[\begin{array}{c} \text{Metóda: substitučná + integrovanie racionálnych funkcií} \\ t = \cos x, dt = -\sin x dx \\ \text{Výsledok } : \operatorname{arctg}\left(\frac{\cos x+1}{2}\right) + C \end{array}\right]$

21. $\int \frac{1}{\sin x}dx.$ $\left[\begin{array}{c} \int \frac{1}{\sin x}dx = \int \frac{\sin x}{\sin^2 x}dx = \int \frac{\sin x}{1-\cos^2 x}dx \\ \text{Metóda: substitučná + integrovanie racionálnych funkcií} \\ t = \cos x, dt = -\sin x dx \\ \text{Výsledok: } \frac{1}{2}\ln\left|\frac{\cos x-1}{\cos x+1}\right| + C \end{array}\right]$

4 Piaty týždeň

1. $\int \frac{2}{x(\ln x-2)(\ln^2 x-2\ln x+2)}dx.$ $\left[\begin{array}{c}\text{Metóda: substitučná+integrovanie racionálnych funkcií}\\ t=\ln x, dt=\frac{1}{x}dx,\\ \text{Výsledok : } \ln\frac{|\ln x-2|}{\left((\ln x-1)^2+1\right)^{\frac{1}{2}}}-\operatorname{arctg}(\ln x-1)+C\end{array}\right]$

2. $\int \frac{1}{\cos x}dx.$ $\left[\begin{array}{c}\text{Metóda: substitučná + integrovanie racionálnych funkcií}\\ t=\operatorname{tg}\frac{x}{2}, x=2\operatorname{arctg}t, dx=\frac{2}{1+t^2}dt, \cos t=\frac{1-t^2}{1+t^2}.\\ \text{Výsledok : }\ln\left|\frac{\operatorname{tg}\frac{x}{2}+1}{\operatorname{tg}\frac{x}{2}-1}\right|+C\end{array}\right]$

3. $\int_1^{64} \frac{2\sqrt{x}}{x\left(\sqrt[3]{x}+\sqrt{x}\right)}dx.$ $\left[\begin{array}{c}\text{Metóda: substitučná}\\ t=\sqrt[6]{x}, x=t^6, dx=6t^5dt, x^{\frac{1}{3}}=t^2, x^{\frac{1}{2}}=t^3.\\ \text{Výsledok: } 12\ln\frac{3}{2}\end{array}\right]$

4. $\int \frac{x}{1+\sqrt{x-1}}dx.$ $\left[\begin{array}{c}\text{Metóda: substitučná+ integrovanie racionálnych funkcií}\\ t=\sqrt{x-1}, x=t^2+1, dx=2tdt.\\ \text{Výsledok: } \frac{2}{3}(x-1)^{\frac{3}{2}}-x+1+4\sqrt{x-1}-4\ln\left(1+\sqrt{x-1}\right)+C\end{array}\right]$

5. $\int \sqrt{\frac{1-x}{1+x}}\frac{dx}{x}.$ $\left[\begin{array}{c}\text{Metóda: substitučná+ integrovanie racionálnych funkcií}\\ t=\sqrt{\frac{1-x}{1+x}}, x=\frac{1-t^2}{1+t^2}, dx=-\frac{4t}{(1+t^2)^2}dt.\\ \text{Výsledok: } \ln\left|\frac{\sqrt{\frac{1-x}{1+x}}-1}{\sqrt{\frac{1-x}{1+x}}+1}\right|+2\operatorname{arctg}\sqrt{\frac{1-x}{1+x}}+C\end{array}\right]$

6. $\int \sqrt{x^2+4x+3}\,dx.$ $\left[\begin{array}{c}\text{Metóda: Eulerova substitúcia}\\ \sqrt{x^2+4x+3}=x-t, x=\frac{t^2-3}{2(t+2)}, dx=\frac{2t^2+8t+6}{4(t+2)^2}dt.\\ \text{Výsledok: } -\frac{1}{8}\left(x-\sqrt{x^2+4x+3}\right)^2-\frac{1}{2}\left(x-\sqrt{x^2+4x+3}\right)+\\ +\frac{1}{2}\ln\left|x-\sqrt{x^2+4x+3}+2\right|+\frac{1}{8\left(x-\sqrt{x^2+4x+3}+2\right)^2}+C\end{array}\right]$

7. $\int \frac{1}{x-\sqrt{x^2-x+1}}\,dx.$ $\left[\begin{array}{c}\text{Metóda: Eulerova substitúcia}\\ \sqrt{x^2-x+1}=x+t, x=\frac{1-t^2}{2t+1}, dx=\frac{-2t^2-2t-2}{(2t+1)^2}dt.\\ \text{Výsledok: } 2\ln\left|x-\sqrt{x^2-x+1}\right|-2\ln\left|2\sqrt{x^2-x+1}-2x+1\right|+\\ +\frac{1}{2\left(\sqrt{x^2-x+1}-2x+1\right)}+C\end{array}\right]$

8. $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{3+\cos x}dx.$ $\left[\begin{array}{c}\text{Metóda: univerzálna trigonometrická substitúcia+}\\ \text{+integrovanie racionálnej funkcie.}\\ t=\operatorname{tg}\frac{x}{2}, x=2\operatorname{arctg}t, dx=\frac{2}{1+t^2}dt, \cos t=\frac{1-t^2}{1+t^2}.\\ x=\frac{\pi}{3}\rightarrow\ t=\frac{1}{\sqrt{3}}, x=\frac{\pi}{2}\rightarrow\ t=1,\\ \text{Výsledok:}\frac{1}{\sqrt{2}}\left[\frac{\pi}{4}-\operatorname{arctg}\frac{1}{\sqrt{6}}\right]\end{array}\right]$

9. $\int \frac{1}{\sin x-\cos x}dx.$ $\left[\begin{array}{c}\text{Metóda: univerzálna trigonometrická substitúcia+}\\ \text{+integrovanie racionálnej funkcie.}\\ t=\operatorname{tg}\frac{x}{2}, x=2\operatorname{arctg} t, dx=\frac{2}{1+t^2}dt,\\ \cos t=\frac{1-t^2}{1+t^2}, \sin t=\frac{2t}{1+t^2}.\\ \text{Výsledok :}\frac{1}{\sqrt{2}}\ln\left|\frac{\operatorname{tg}\frac{x}{2}+1-\sqrt{2}}{\operatorname{tg}\frac{x}{2}+1+\sqrt{2}}\right|+C\end{array}\right]$

10. $\int \frac{1-\sin x}{1+\cos x}dx.$ $\left[\begin{array}{c}\text{Metóda: univerzálna trigonometrická substitúcia+}\\ \text{+integrovanie racionálnej funkcie.}\\ t=\operatorname{tg}\frac{x}{2}, x=2\operatorname{arctg} t, dx=\frac{2}{1+t^2}dt,\\ \cos t=\frac{1-t^2}{1+t^2}, \sin t=\frac{2t}{1+t^2}.\\ \text{Výsledok : }\operatorname{tg}\frac{x}{2}-\ln\left(\operatorname{tg}^2\frac{x}{2}+1\right)+C\end{array}\right]$

11. $\int_0^{\frac{\pi}{2}} \frac{1}{3-2\sin x+\cos x}dx.$ $\left[\begin{array}{c}\text{Metóda: univerzálna trigonometrická substitúcia+}\\ \text{+integrovanie racionálnej funkcie.}\\ t=\operatorname{tg}\frac{x}{2}, x=2\operatorname{arctg} t, dx=\frac{2}{1+t^2}dt,\\ \cos t=\frac{1-t^2}{1+t^2}, \sin t=\frac{2t}{1+t^2}.\\ x=0\rightarrow\ t=0, x=\frac{\pi}{2}\rightarrow\ t=1,\\ \text{Výsledok :}\frac{\pi}{4}\end{array}\right]$

12. $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1+\operatorname{tg}^2 x}{(1+\operatorname{tg} x)^2}dx.$ $\left[\begin{array}{c}\text{Metóda:trigonometrická substitúcia+integrovanie racionálnej funkcie.}\\ t=\operatorname{tg} x, x=\operatorname{arctg} t, dx=\frac{1}{1+t^2}dt,\\ x=\frac{\pi}{4}\rightarrow\ t=1, x=\frac{\pi}{3}\rightarrow\ t=\sqrt{3}.\\ \text{Výsledok :}\frac{2-\sqrt{3}}{2}\end{array}\right]$

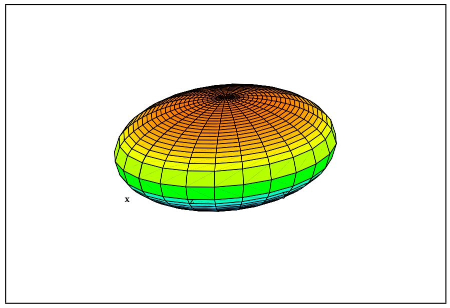
13. $\int_0^{\ln 5} \frac{e^x\sqrt{e^x-1}}{e^x+3}dx.$ $\left[\begin{array}{c}\text{Metóda: substitučná + integrovanie racionálnych funkcií}\\ t=e^x-1, dt=e^x dx, x=0\rightarrow\ t=0, x=\ln 5\rightarrow t=4\\ s=\sqrt{t}, t=s^2, dt=2sds, t=0\rightarrow s=0, t=4\rightarrow s=2\\ \text{Výsledok : }4-\pi\end{array}\right]$

14. $\int \frac{e^x+10}{e^{2x}-2e^x+5}dx.$ $\left[\begin{array}{c}\text{Metóda: substitučná+integrovanie racionálnych funkcií}\\ \text{Výsledok : }2x-\ln\left(e^{2x}-2e^x+5\right)+\frac{3}{2}\operatorname{arctg}\left(\frac{1}{2}e^x-\frac{1}{2}\right)+C\end{array}\right]$

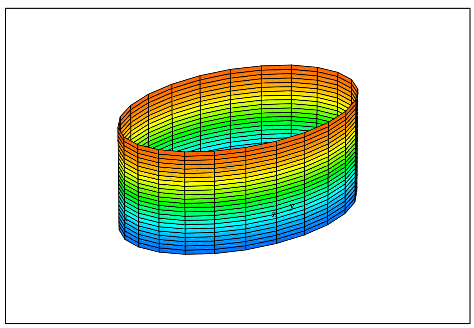
15. $\int x^2e^{\sqrt{x}}dx.$ $\left[\begin{array}{c}\text{Metóda: substitúcia + per partes}\\ \text{Výsledok : }\left[2\left(\sqrt{x}\right)^5-10x^2+40\left(\sqrt{x}\right)^3-120x+240\sqrt{x}-240\right]e^{\sqrt{x}}+C\end{array}\right]$

5 Kvadratické plochy

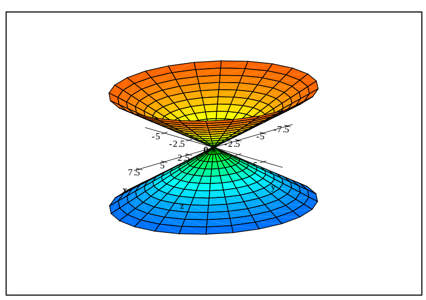
Elipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, $\left(\frac{x^2}{9} + \frac{y^2}{6} + \frac{z^2}{4} = 1\right)$



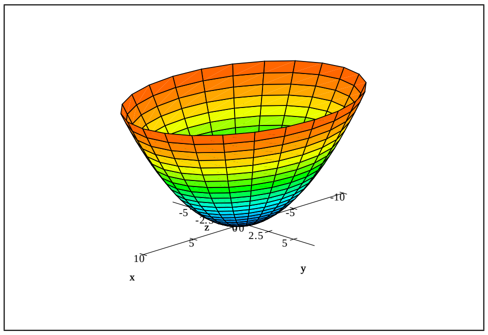
Eliptická valcová plocha $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $\left(\frac{x^2}{4} + \frac{y^2}{3} = 1\right)$



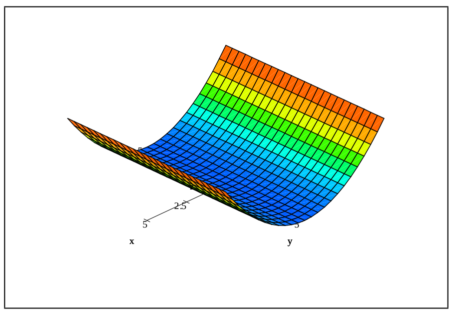
Eliptická kužeľová plocha $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$, $\left(\frac{x^2}{4} + \frac{y^2}{3} - \frac{z^2}{4} = 0\right)$



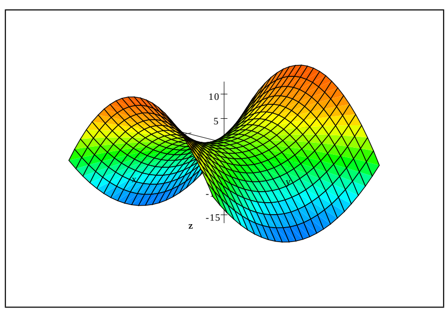
Eliptický paraboloid $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z}{c} = 0$, $\left(\frac{x^2}{4} + \frac{y^2}{3} - \frac{z}{1} = 0\right)$



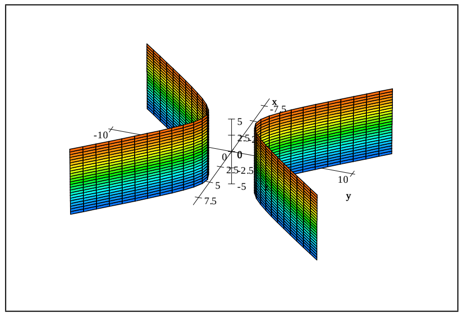
Parabolická valcová plocha $z = ax^2$, $\left(z = \frac{1}{4}x^2\right)$



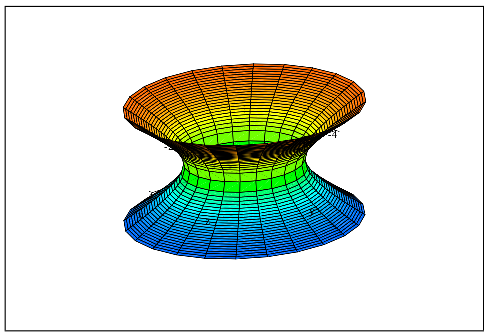
Hyperbolický paraboloid $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z}{c} = 0$, $\left(\frac{x^2}{4} - \frac{y^2}{3} - \frac{z}{2} = 0\right)$



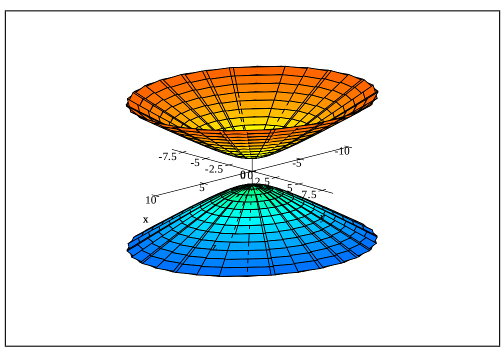
Hyperbolická valcová plocha $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$, $\left(\frac{y^2}{4} - \frac{x^2}{3} = 1\right)$



Jednodielny hyperboloid $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$, $\left(\frac{x^2}{4} + \frac{y^2}{3} - \frac{z^2}{5} = 1\right)$



Dvojdielny hyperboloid $\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, $\left(\frac{z^2}{5} - \frac{x^2}{4} - \frac{y^2}{3} = 1\right)$



6 Šiesty týždeň

Nájdite definičný obor daných funkcií (aj načrtnite):

1. $f(x,y) = \sqrt{x^2 + y^2 - r^2}$, kde $r \geq 0$ je reálne číslo. $\left[D(f) = \left\{(x,y) \in \boldsymbol{R}^2;\ x^2 + y^2 \geq r^2\right\}\right]$
2. $g(x,y) = \frac{1}{\sqrt{r^2 - x^2 - y^2}}$, kde $r \geq 0$ je reálne číslo. $\left[D(g) = \left\{(x,y) \in \boldsymbol{R}^2;\ x^2 + y^2 < r^2\right\}\right]$
3. $f(x,y) = \ln(-x-y)$. $\left[D(f) = \left\{(x,y) \in \boldsymbol{R}^2;\ x < -y\right\}\right]$
4. $f(x,y,z) = \ln\left(1 - x^2 - y^2 + z^2\right)$. $\left[D(f) = \left\{(x,y,z) \in \boldsymbol{R}^3;\ x^2 + y^2 - z^2 < 1\right\}\right]$
5. $f(x,y,z) = \arccos(2x-1) + \sqrt{1-y^2} + \sqrt{y} + \ln\left(4 - z^2\right)$.
   $\left[D(f) = \left\{(x,y,z) \in \boldsymbol{R}^3;\ 0 \leq x \leq 1,\ 0 \leq y \leq 1,\ -2 < z < 2\right\}\right]$
6. $f(x,y) = \sqrt{4 - x^2 - y^2 + 2x - 4y}$. $\left[D(f) = \left\{(x,y) \in \boldsymbol{R}^2;\ (x-1)^2 + (y+2)^2 \leq 9\right\}\right]$

   Vypočítajte limity:
7. $\lim_{(x,y)\longrightarrow(0,2)} \frac{3y^2 - 3xy - 6y}{1 - \sqrt{x - y + 3}}$. [12]
8. $\lim_{(x,y)\longrightarrow(3,4)} \frac{4 - \sqrt{x + 3y + 1}}{15 - x - 3y}$. $\left[\frac{1}{8}\right]$
9. $\lim_{(x,y)\longrightarrow(-2,1)} \frac{(2x+y)^2 - 9}{4xy + 2y^2 + 6y}$. $[-3]$
10. $\lim_{(x,y)\longrightarrow(4,0)} \frac{\operatorname{tg}(xy)}{y}$. [4]
11. $\lim_{(x,y)\longrightarrow(0,0)} \frac{2 - \sqrt{4 - xy}}{xy}$. $\left[\frac{1}{4}\right]$
12. $\lim_{(x,y,z)\longrightarrow(1,1,1)} \frac{\sin(x+y-z-1)}{x+y-z-1}$. [1]
13. $\lim_{(x,y)\longrightarrow(2,3)} \frac{\sin x + 2}{(x^2 - y^2 + 5)^2}$. $[+\infty]$
14. $\lim_{(x,y)\longrightarrow(0,0)} \frac{2xy}{xy + 2x - y}$. [neexistuje]
15. $\lim_{(x,y)\longrightarrow(0,0)} \frac{xy^2}{x^2 + y^4}$. [neexistuje]
16. Dodefinujte funkciu $f(x,y) = \frac{xy}{3 - \sqrt{xy + 9}}$ tak, aby bola v bode $(0,0)$ spojitá.
    $[f(0,0) = -6]$
17. Dodefinujte funkciu $f(x,y) = \frac{x^3 - y^3}{x^4 - y^4}$ tak, aby bola v bode $(2,2)$ spojitá.
    $\left[f(2,2) = \frac{3}{8}\right]$
18. Dodefinujte funkciu $f(x,y) = \frac{x^2 y^2}{x^2 y^2 + x - y}$ tak, aby bola v bode $(0,0)$ spojitá.
    [Funkcia sa nedá dodefinovať v bode $(0,0)$ aby bola spojitá]

19. Daná je funkcia $f : \boldsymbol{R}^2 \longrightarrow \boldsymbol{R}$ predpisom

$$f(x,y) = \begin{cases} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} & (x,y) \neq (0,0) \\ 2 & (x,y) = (0,0) \end{cases}.$$

Zistite, či je v bode $(0,0)$ spojitá. [Je spojitá]

20. Daná je funkcia $f : A = \left\{\mathbf{x} \in \boldsymbol{R}^2 : y \neq 0\right\} \longrightarrow \boldsymbol{R}$, $f(x,y) = \frac{\sin(6xy)}{y}$. Dodefinujte funkciu v bode $(3,0)$ tak, aby bola v tomto bode spojitá. spojitá. $[f(3,0) = 18]$

21. Daná je funkcia $f : \boldsymbol{R}^2 \longrightarrow \boldsymbol{R}$ predpisom

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}.$$

Zistite, či je f spojitá. [Je spojitá všade s výnimkou bodu $(0,0)$]

7 Siedmy týždeň

1. Zistite, či je funkcia $f : \boldsymbol{R}^2 \longrightarrow \boldsymbol{R},\ f(x,y) = \begin{cases} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} & (x,y) \neq (0,0) \\ 1 & (x,y) = (0,0) \end{cases}$ diferencovateľná v bode $(0,0)$.

   [nie je]

2. Zistite, či je funkcia $f : \boldsymbol{R}^2 \longrightarrow \boldsymbol{R},\ f(x,y) = \sqrt{|xy|}$ diferencovateľná v bode $(0,0)$.

   [nie je]

3. Daná je funkcia $f : \boldsymbol{R}^3 \longrightarrow \boldsymbol{R},\ f(x,y,z) = \begin{cases} \frac{2x-3y+z^2}{\sqrt{x^2+y^2+z^2}} & (x,y,z) \neq (0,0,0) \\ 0 & (x,y,z) = (0,0,0) \end{cases}$ .Vypočítajte

   (a) parciálne derivácie v bode $(0,0,0)$,

   (b) zistite, či je funkcia v bode $(0,0,0)$ diferencovateľná. $\left[\frac{\partial f}{\partial x}(0,0,0) = \infty,\ \frac{\partial f}{\partial y}(0,0,0) = -\infty,\ \frac{\partial f}{\partial z}(0,0,0)\right.$

4. Pomocou definície vypočítajte parciálne derivácie funkcie $f(x,y) = \left(x^2+y\right)\sin(x+y)$ v bode $\boldsymbol{a} = (0,\pi)$.

   $\left[\frac{\partial f}{\partial x}(0,\pi) = -\pi,\ \frac{\partial f}{\partial y}(0,\pi) = -\pi\right]$

5. Pomocou definície vypočítajte parciálne derivácie funkcie $f(x,y) = 4x^3 - 2y^2 + 3xy^2 + 5y$ v bode $\boldsymbol{a} = (1,2)$.

   $\left[\frac{\partial f}{\partial x}(1,2) = 24,\ \frac{\partial f}{\partial y}(1,2) = 9\right]$

6. Pomocou definície vypočítajte parciálne derivácie funkcie $f(x,y) = \begin{cases} \left(x^2+y^2\right)\cos\left(\frac{1}{x^2+y^2}\right) & (x,y) \neq (0 \\ 0 & (x,y) = (0 \end{cases}$ v bode $\boldsymbol{a} = (0,0)$.

   $\left[\frac{\partial f}{\partial x}(0,0) = 0,\ \frac{\partial f}{\partial y}(0,0) = 0\right]$

7. Vypočítajte $\frac{\partial f(x,y)}{\partial x}$ a $\frac{\partial^2 f(x,y)}{\partial y \partial x}$ keď $f(x,y) = \begin{cases} \frac{2x^3}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$.

   $\left[\frac{\partial f(x,y)}{\partial x} = \frac{2x^4+6x^2y^2}{(x^2+y^2)^2},\ \frac{\partial f(0,0)}{\partial x} = 2,\ \frac{\partial^2 f(x,y)}{\partial y \partial x} = \frac{4x^4y-12x^2y^3}{(x^2+y^2)^3},\ \frac{\partial^2 f(0,0)}{\partial y \partial x} - \text{neexistuje}\right]$

8. Nech $f : \boldsymbol{R}^2 \longrightarrow \boldsymbol{R},\ f(x,y) = \begin{cases} \left(x^2+y^2\right)\sin\frac{1}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$.

   Vypočítajte $\frac{\partial^2 f(0,0)}{\partial y \partial x}$, $\frac{\partial^2 f(0,0)}{\partial x \partial y}$.

   $\left[\frac{\partial^2 f(0,0)}{\partial y \partial x} = 0,\ \frac{\partial^2 f(0,0)}{\partial x \partial y} = 0\right]$

9. Vypočítajte rovnicu dotykovej roviny a normály ku grafu funkcie $f(x,y) = xy$ v bode $T = (?,2,2)$.

   $[\tau : 2x + y - z - 2 = 0, n : x = 1 + 2t, y = 2 + t, z = 2 - t.]$

10. Vypočítajte rovnicu dotykovej roviny a normály ku grafu funkcie $f(x,y) = 2x^2 + y^2$ v bode $T = (1,1,?)$.

    $[\tau : 4x + 2y - z - 3 = 0, n : x = 1 + 4t,\ y = 1 + 2t,\ z = 3 - t\,]$

11. Vypočítajte parciálne derivácie, gradient a diferenciál funkcie $f(x,y) = \frac{2}{(3x^2+4y^2)^2}$ v bode $\boldsymbol{a} = (-1,1)$. $\left[\begin{array}{c} \frac{\partial f}{\partial x}(-1,1) = \frac{24}{343},\ \frac{\partial f}{\partial y}(-1,1) = -\frac{32}{343},\ grad\ f(-1,1) = \left(\frac{24}{343}, -\frac{32}{343}\right), \\ \mathcal{D}f(-1,1)(\boldsymbol{h}) = \frac{24}{343}h_1 - \frac{32}{343}h_2, \text{ kde } \boldsymbol{h} = (h_1,h_2) \end{array}\right]$

12. Vypočítajte $\frac{\partial f(x,y)}{\partial x}$ a $\frac{\partial^2 f(x,y)}{\partial y \partial x}$ keď $f(x,y) = \begin{cases} \frac{2x^3}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$.

    $\left[\frac{\partial f(x,y)}{\partial x} = \frac{2x^4+6x^2y^2}{(x^2+y^2)^2},\ \frac{\partial f(0,0)}{\partial x} = 2,\ \frac{\partial^2 f(x,y)}{\partial y \partial x} = \frac{4x^4y-12x^2y^3}{(x^2+y^2)^3},\ \frac{\partial^2 f(0,0)}{\partial y \partial x} - \text{neexistuje}\right]$

13. Nech $f : \boldsymbol{R}^2 \longrightarrow \boldsymbol{R}$, $f(x,y) = \begin{cases} \left(x^2+y^2\right)\sin\frac{1}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$.

    Vypočítajte $\frac{\partial^2 f(0,0)}{\partial y \partial x}$, $\frac{\partial^2 f(0,0)}{\partial x \partial y}$.

    $\left[\frac{\partial^2 f(0,0)}{\partial y \partial x} = 0,\ \frac{\partial^2 f(0,0)}{\partial x \partial y} = 0\right]$

14. Vypočítajte deriváciu funkcie $f(x,y) = e^y \cos(x+y)$ v bode $\boldsymbol{a} = \left(\frac{\pi}{2}, 0\right)$ v smere vektora $\boldsymbol{e} = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

    $\left[\frac{df}{d\boldsymbol{e}}(\boldsymbol{a}) = f_{.\boldsymbol{e}}(\boldsymbol{a}) = -\frac{1}{2}\left(1+\sqrt{3}\right)\right]$

15. Nájdite deriváciu funkcie $f(x,y) = 3x^2 - 6xy + y^2$ v bode $\boldsymbol{a} = \left(-\frac{1}{3}, -\frac{1}{2}\right)$ v smere ľubovoľného jednotkového vektora $\boldsymbol{e}$. Zistite v akom smere je derivácia

    (a) nulová, $\left[\boldsymbol{e}_1 = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right),\ \boldsymbol{e}_2 = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)\right]$

    (b) najväčšia, $\left[\boldsymbol{e}_3 = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)\right]$

    (c) najmenšia. $\left[\boldsymbol{e}_4 = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)\right]$

8 Ôsmy týždeň

V nasledujúcich príkladoch nájdite lokálne extrémy funkcií

1. $f(x,y) = 2x^3 - xy^2 + 5x^2 + y^2$. $\left[f(0,0) = 0\right.$ lokálne minimum, sedlové body $(1,4)$, $(1,-4)$, $\left.\left(-\frac{5}{3}, 0\right)\right]$
2. $f(x,y) = e^{2x}\left(x + y^2 + 2y\right)$.
   $\left[f\left(\frac{1}{2}, -1\right) = -\frac{e}{2}\right.$ lokálne minimum$\left.\right]$
3. $f(x,y) = x^3 + y^3 + 3xy + 2$.
   $[f(-1,-1) = 3$ lokálne maximum, sedlový bod $(0,0)]$
4. $f(x,y) = 5xy + \frac{25}{x} + \frac{8}{y}$, $x > 0$, $y > 0$.
   $\left[f\left(\frac{5}{2}, \frac{4}{5}\right) = 30\right.$ lokálne minimum$\left.\right]$
5. $f(x,y) = \frac{1}{2}y + (47 - x - y)\left(\frac{x}{3} + \frac{y}{4}\right)$.
   $[(-99, 140)$ sedlový bod$]$
6. $f(x,y) = xy(2 - x - y)$.
   $\left[f\left(\frac{2}{3}, \frac{2}{3}\right) = \frac{8}{27}\right.$ lokálne maximum , sedlové body $\left.(0,0), (0,2), (2,0)\right]$
7. $f(x,y) = e^{-x^2-y^2}\left(2y^2 + x^2\right)$. $[f(0,0) = 0$ lokálne minimum, $f(0,1) = \frac{2}{e}$, $f(0,-1) = \frac{2}{e}$ lokálne maxim
8. $f(x,y) = x^2y^2(3 - 4x + 6y)$.
   $\left[\begin{array}{c} f\left(\frac{3}{10}, -\frac{1}{5}\right) = \frac{27}{12400} \text{ lokálne maximum,} \\ \text{v bodoch } \left\{(x,0) : x > \frac{3}{4}\right\} \wedge \left\{(0,y) : y < \frac{1}{2}\right\} \text{ sú lokálne maximá, pre ktoré } f(.,.) = 0, \\ \text{v bodoch } \left\{(x,0) : x < \frac{3}{4}\right\} \wedge \left\{(0,y) : y > \frac{1}{2}\right\} \text{ sú lokálne minimá, pre ktoré } f(.,.) = 0, \\ \text{sedlové body } \left(\frac{3}{4}, 0\right) a \left(0, \frac{1}{2}\right). \end{array}\right]$
9. $f(x,y,z) = x^2 + y^2 + z^2 - xy + 2z + x$.
   $\left[f\left(-\frac{2}{3}, -\frac{1}{3}, -1\right) = -\frac{4}{3}\right.$ lokálne minimum$\left.\right]$
10. $f(x,y,z) = 3x^2 + 3x + 2y^2 + 2yz + 2y + 2z^2 - 2z$.
    $\left[f\left(-\frac{1}{2}, -1, 1\right) = -\frac{11}{4}\right.$ lokálne minimum$\left.\right]$
11. $f(x,y,z) = 2x^2 + y^2 + 2z - xy - xz$.
    $[(2,1,7)$ - sedlový bod$]$
12. $f(x,y,z) = x^2 + y^2 + z^2 + 2x + 4y - 6z$.
    $[f(-1,-2,3) = -14$ lokálne minimum$]$
13. $f(x,y,z) = y^2 + 2z^2 + 2x - xy - xz$.
    $\left[\left(\frac{8}{3}, \frac{4}{3}, \frac{2}{3}\right)\right.$ - sedlový bod$\left.\right]$
14. $f(x,y,z) = x^3 + y^2 + z^2 + 12xy + 2z$. $[f(24, -144, -1) = -6913$ lokálne minimum, $(0,0,-1)$ - sedlový bod

9 Deviaty týždeň

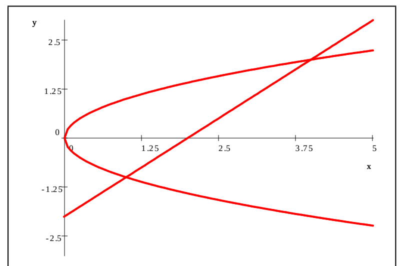
Vypočítajte dvojné integrály:

1. $\iint_I \frac{x}{(1+x^2+y^2)^{\frac{3}{2}}} dxdy,\ I = \langle 0,1\rangle \times \langle 0,1\rangle$. $\left[\ln \frac{2+\sqrt{2}}{1+\sqrt{3}}\right]$

2. $\iint_I \frac{1}{(1-xy)^2} dxdy,\ I = \langle 2,3\rangle \times \langle 1,2\rangle$. $\left[\ln\left(\frac{6}{5}\right)\right]$

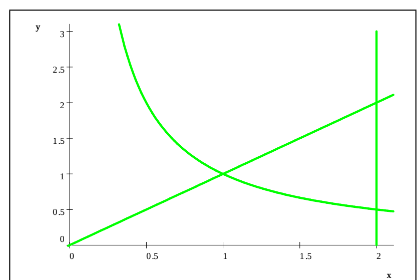
   Vypočítajte dvojné integrály, načrtnite obrázok množiny A : (Na nasledujúcich obrázkoch sú pre Vašu lepšiu predstavu nakreslené odpovedajúce hranice elementárnych oblastí.)

3. $\iint_A ye^x dxdy,\ A = \left\{(x,y)\,;\, y^2 \le x \le y+2\right\}$.



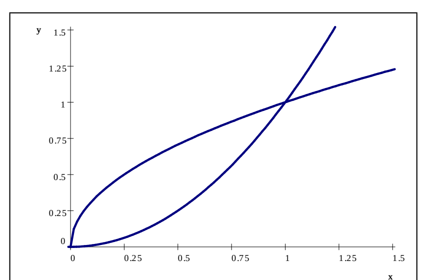
$\left[\frac{1}{2}\left(e^4 + 5e\right)\right]$

4. $\iint_A \frac{x^2}{y^2} dxdy,\ A = \left\{(x,y)\,;\, 0 \le \frac{1}{x} \le y \le x,\ x \le 2\right\}$.



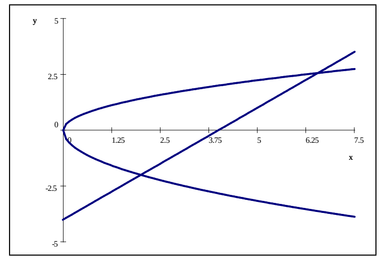
$\left[\frac{9}{4}\right]$

5. $\iint_A \left(3x^2 + 2y\right) dxdy,\ A = \left\{(x,y)\,;\, x^2 \le y \le \sqrt{x},\ x \ge 0\right\}$.



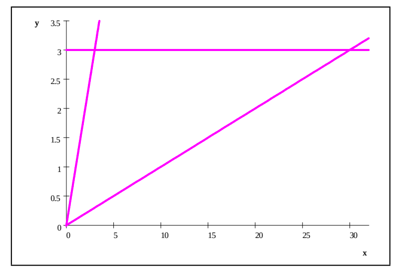
$\left[\frac{39}{70}\right]$

6. $\iint_A xydxdy,\ A = \left\{(x,y)\,;\, x-4 \le y,\ y^2 \le 2x\right\}.$



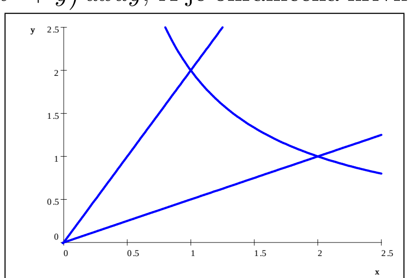
[90]

7. $\iint_A \sqrt{xy-y^2}dxdy,\ A = \left\{(x,y)\,;\, 0 \le y \le 3,\ \frac{x}{10} \le y \le x\right\}.$



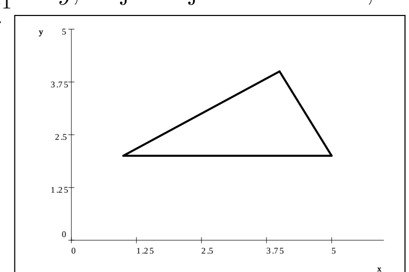
[162]

8. $\iint_A \left(x^2+y\right)dxdy,\ A$ je ohraničená krivkami $y=\frac{1}{2}x,\ y=2x,\ xy=2,\ x \ge 0.$



[$\frac{17}{6}$]

9. $\iint_A \frac{1}{x+y+1}dxdy,\ A$ je trojuholník $KLM,\ K=(1,2),\ L=(5,2),\ M=(4,4).$

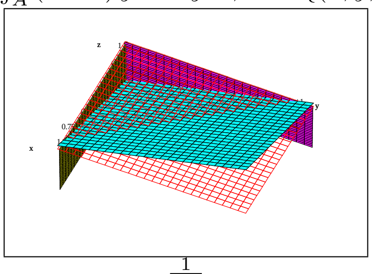


[$\frac{72}{5}\ln 18 - 16\ln 16 + \frac{8}{5}\ln 8$]

10 Desiaty týždeň

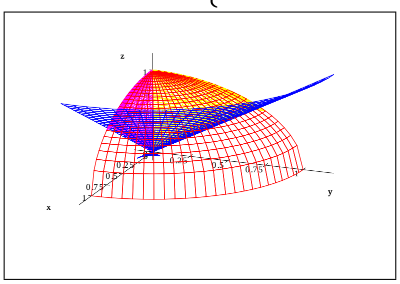
Vypočítajte trojné integrály. Načrtnite obrázok množiny A.

1. $\iiint_A (1-x)\,yz\,dxdydz,\ A = \{(x,y,z)\,;\, x \ge 0,\ y \ge 0,\ z \ge 0,\ z \le 1-x-y\}$.

[

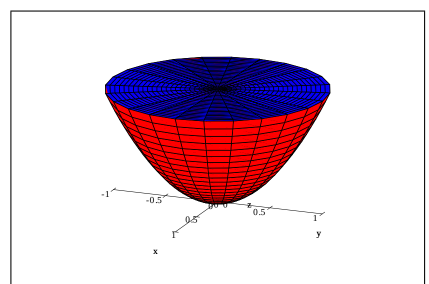
$\frac{1}{144}$]

2. $\iiint_A z\,dxdydz,\ A = \left\{(x,y,z)\,;\, x \ge 0,\ y \ge 0,\ \sqrt{x^2+y^2} \le z \le \sqrt{1-x^2-y^2}\right\}$.

[

$\frac{\pi}{8}$]

3. $\iiint_A \left(x^2+y^2\right) dxdydz,\ A = \left\{(x,y,z)\,;\, x^2+y^2 \le 2z,\ z \le 2\right\}$.

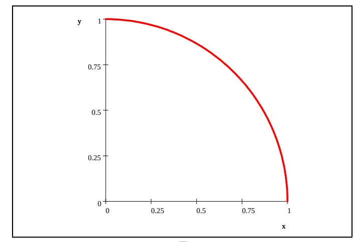
[

$\frac{16}{3}\pi$]

11 Jedenásty týždeň

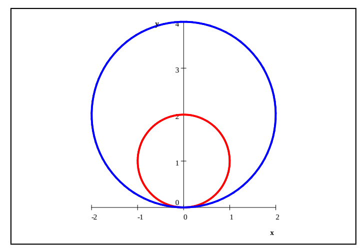
V príkladoch 1 - 8 vypočítajte dvojné integrály použitím vhodnej transformácie:

1. $\iint_A \sqrt{1-x^2-y^2}\,dxdy,\ A=\left\{(x,y)\,;\, x^2+y^2\le 1,\ x\ge 0,\ y\ge 0\right\}.$



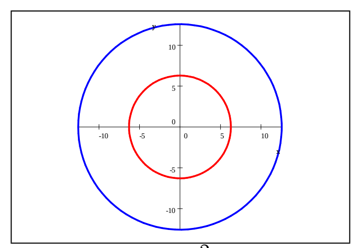
$\left[\ \frac{\pi}{6}\ \right]$

2. $\iint_A xy^2\,dxdy,\ A=\left\{(x,y)\,;\, 2y\le x^2+y^2\le 4y\right\}.$



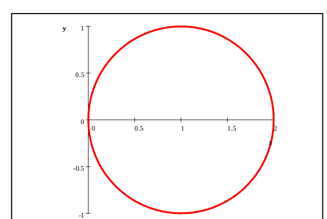
$\left[\ 0\ \right]$

3. $\iint_A \sin\sqrt{x^2+y^2}\,dxdy,\ A=\left\{(x,y)\,;\, \pi^2\le x^2+y^2\le 4\pi^2\right\}.$



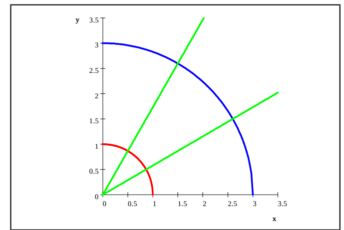
$\left[\ -6\pi^2\ \right]$

4. $\iint_A \sqrt{a^2-x^2-y^2}\,dxdy,\ A=\left\{(x,y)\,;\, x^2+y^2\le ax\right\}.$



$\left[\ \frac{a^3}{3}\left(\pi-\frac{4}{3}\right)\ \right]$

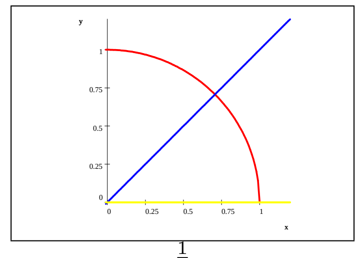
5. $\iint_A \operatorname{arctg} \frac{y}{x} dxdy,\ A = \left\{(x,y)\,;\, 1 \le x^2+y^2 \le 9,\ \frac{x}{\sqrt{3}} \le y \le \sqrt{3}x\right\}.$ [



$\frac{\pi^2}{6}$]

6. $\iint_A (1-2x-3y)\,dxdy,\ A = \{(x,y)\,;\, x^2+y^2 \le 4\}.$ $[4\pi]$

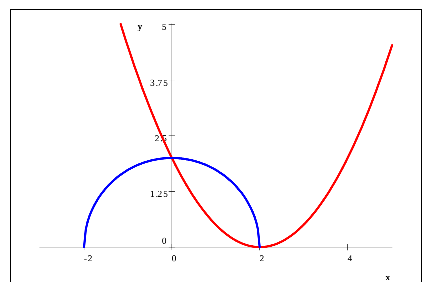
7. $\iint_A \sin\left(\pi\sqrt{x^2+y^2}\right) dxdy,\ A = \{(x,y)\,;\, x^2+y^2 \le 1,\ 0 \le x \le y\}.$ [



$\frac{1}{4}$]

8. $\iint_A \ln\left(1+x^2+y^2\right) dxdy,\ A = \left\{(x,y)\,;\, y \le \sqrt{R^2-x^2},\ x \ge 0,\ y \ge 0\right\}.$
$\left[\frac{\pi}{4}\left(1+R^2\right)\ln\left(1+R^2\right) - R^2\right]$
V úlohách 9 – 11 vypočítajte plošný obsah rovinných obrazcov určených množinou A, keď

9. A je ohraničená krivkami: $y = \frac{1}{2}(x-2)^2$ a $x^2+y^2 = 4.$ [



$\pi - \frac{4}{3}$]

Priesečník kriviek: $y = \frac{1}{2}(x-2)^2$ a $x^2+y^2 = 4 : x^2 + \left(\frac{1}{2}(x-2)^2\right)^2 = 4 \Rightarrow$

$\Rightarrow x^2 - 4 + \left(\frac{1}{2}(x-2)^2\right)^2 = 0 \Rightarrow 4(x-2)(x+2) + (x-2)^4 = 0 \Rightarrow$

$\Rightarrow (x-2)\left[4(x+2) + (x-2)^3\right] = 0 \Rightarrow (x-2)\left[4x+8+x^3-6x^2+12x-8\right] = 0 \Rightarrow$

$\Rightarrow\ x(x-2)\left[x^2-6x+16\right] = 0 \Rightarrow x_1 = 0, x_2 = 2,$ ostatné korene sú komplexné

$A : 0 \le x \le 2, \frac{1}{2}(x-2)^2 \le y \le \sqrt{4-x^2}$

$$P = \iint\limits_A 1dxdy = \int\limits_0^2 \left(\int\limits_{\frac{1}{2}(x-2)^2}^{\sqrt{4-x^2}} 1dy \right) dx = \int\limits_0^2 \left(\sqrt{4-x^2} - \frac{1}{2}(x-2)^2 \right) dx =$$

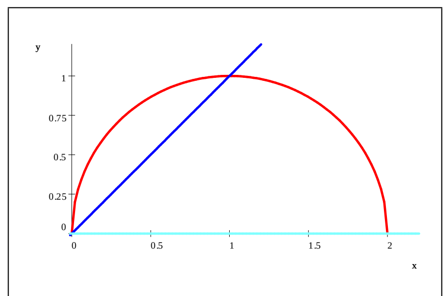
$$= \int\limits_0^2 \sqrt{4-x^2}dx - \frac{1}{2}\int\limits_0^2 (x-2)^2 dx = \left| \begin{array}{c} x = 2\sin t \\ dx = 2\cos t dt \\ x = 0 \leftrightarrow t = 0 \\ x = 2 \leftrightarrow t = \frac{\pi}{2} \end{array} \right| = \int\limits_0^{\frac{\pi}{2}} \sqrt{4 - 4\sin^2 t} 2\cos t dt -$$

$$\frac{1}{2}\left[\frac{(x-2)^3}{3}\right]_0^2 =$$

$$= 4\int\limits_0^{\frac{\pi}{2}} \cos^2 t dt - \frac{1}{6}\left(0 - (-2)^3\right) dx = 4\frac{1}{2}\int\limits_0^{\frac{\pi}{2}} (1 + \cos 2t)\, dt - \frac{4}{3} = 2\left[t + \frac{\sin 2t}{2}\right]_0^{\frac{\pi}{2}} -$$

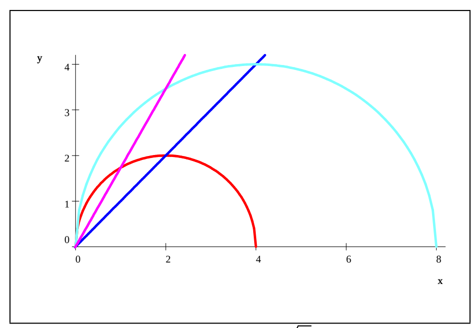
$$\frac{4}{3} = \pi - \frac{4}{3}$$

10. $A = \left\{(x,y)\,;\, 0 \le y \le x,\, x^2 + y^2 \le 2x\right\}.$



$\frac{\pi}{4} + \frac{1}{2}$

11. $A = \left\{(x,y)\,;\, x \le y \le \sqrt{3}x,\, 4x \le x^2 + y^2 \le 8x\right\}.$

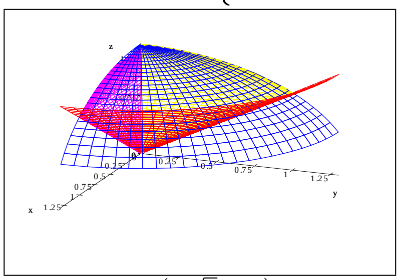


$\pi - 6 + 3\sqrt{3}$

12 Dvanásty týždeň

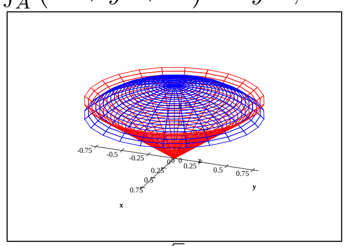
V úlohách 1 – 5 vypočítajte trojné integrály použitím vhodnej transformácie. Načrtnite obrázok množiny A.

1. $\iiint_A z^2 dxdydz,\ A = \left\{(x,y,z)\,;\ x \geq 0,\ y \geq 0,\ \sqrt{x^2+y^2} \leq z \leq \sqrt{2-x^2-y^2}\right\}.$



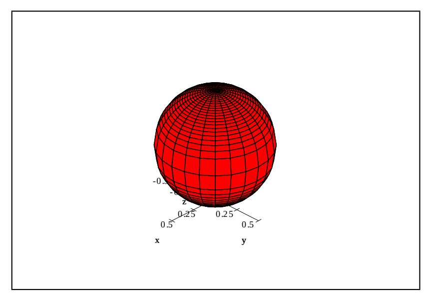
$\left[\ \frac{\pi}{15}\left(2\sqrt{2}-1\right)\ \right]$

2. $\iiint_A \left(x^2+y^2+z^2\right) dxdydz,\ A = \left\{(x,y,z)\,;\ x^2+y^2 \leq z^2,\ x^2+y^2+z^2 \leq R^2\right\}.$



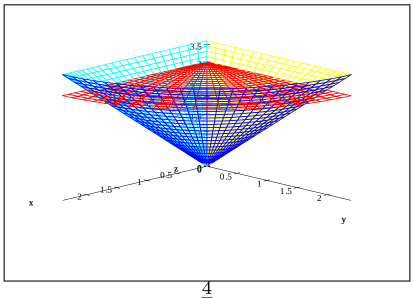
$\left[\ 2\frac{2-\sqrt{2}}{5}\pi R^5\ \right]$

3. $\iiint_A \sqrt{x^2+y^2+z^2}dxdydz,\ A = \left\{(x,y,z)\,;\ x^2+y^2+z^2 \leq z\right\}.$



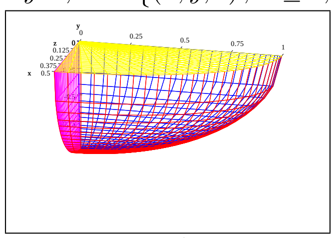
$\left[\ \frac{\pi}{10}\ \right]$

4. $\iiint_A \frac{xy}{\sqrt{z}}dxdydz,\ A = \left\{(x,y,z)\,;\ x \geq 0,\ y \geq 0,\ 0 \leq z \leq 3,\ \frac{x^2}{4}+\frac{y^2}{4}-\frac{z^2}{9} \leq 0\right\}.$



$\left[\dfrac{4}{3} \right]$

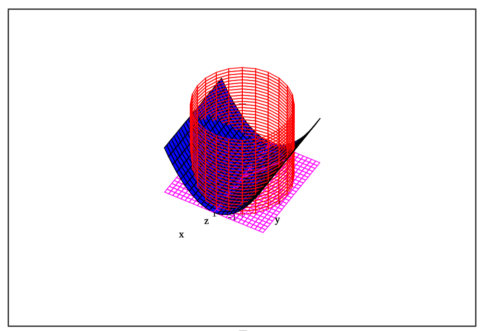
5. $\iiint_A x^2yzdxdydz$, $A = \left\{(x, y, z)\,;\, x \geq 0,\, y \geq 0,\, z \leq 0,\, 4x^2 + y^2 + z^2 \leq 1\right\}.$



[Použite zovšeobecnené sférické súradnice $- \frac{1}{840}$]

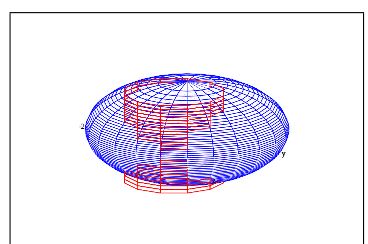
V úlohách 6 – 10 nájdite objem množiny A. Načrtnite obrázok!

6. $A = \left\{(x, y, z)\,;\, x^2 + y^2 \leq 1,\, 0 \leq z \leq y^2\right\}.$



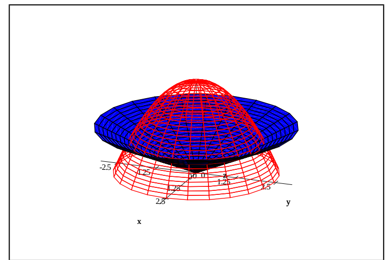
$\left[\dfrac{\pi}{4} \right]$

7. $A = \left\{(x, y, z)\,;\, x^2 + y^2 \leq 2x,\, x^2 + y^2 + z^2 \leq 4\right\}.$



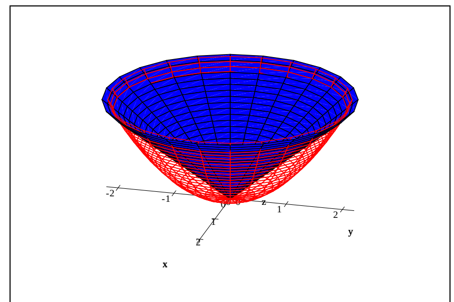
$\left[\frac{16}{9}\,(3\pi - 4) \right]$

8. A je ohraničená plochami $z = 6 - x^2 - y^2$ a $z = \sqrt{x^2 + y^2}$.



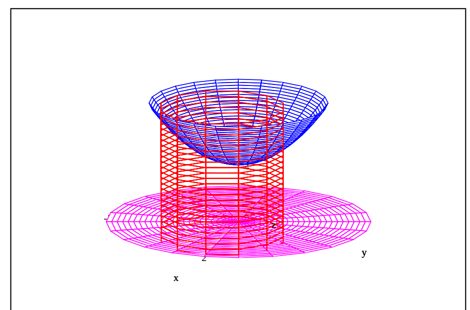
$\left[\frac{32}{3}\pi \right]$

9. A je ohraničená plochami $2z = x^2 + y^2$, $z = \sqrt{x^2 + y^2}$.



$\left[\frac{4}{3}\pi \right]$

10. A je ohraničená plochami $x^2 + y^2 = 2x$, $x^2 + y^2 = z - 2$, $z = 0$.



$\left[\frac{7}{2}\pi \right]$