Príklad 1. [20]

Je daná funkcia

$$f(x,y) = \frac{xy}{x^2+y^2}.$$

a, Vypočítajte rovnicu dotykovej roviny v bode $a = [1,2]$.

b, Vypočítajte deriváciu $\frac{\partial f}{\partial \vec{e}}(1,2)$, ak smer $\vec{e}$ je daný vektorom $\vec{u} = (1, \sqrt{3})$.

c, Vypočítajte limitu (ak existuje)

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$$

$D_f = R^2 \setminus (0,0)$

a) $f'_x = \dfrac{y(x^2+y^2) - 2x^2y}{(x^2+y^2)^2}$ $\quad f'_y = \dfrac{x(x^2+y^2) - 2xy^2}{(x^2+y^2)^2}$ na $R^2 \setminus \{0,0\}$

na $R^2 \setminus (0,0)$ parc. der. spojité $\to$ f diferencovateľná, má dotyk. rovinu

$a = [1,2]$

$$z - f(1,2) = f'_x(1,2)(x-1) + f'_y(1,2)(y-2)$$

$$f'_x(1,2) = \frac{2(1+4) - 2\cdot 1\cdot 2}{(1+4)^2} = \frac{6}{25}$$

$$f'_y(1,2) = \frac{1\cdot(1+4) - 2\cdot 1\cdot 4}{(1+4)^2} = \frac{-3}{25}$$

dotyk. rovina: $\quad z - \frac{2}{5} = \frac{6}{25}(x-1) - \frac{3}{25}(y-2)$

b) e určený $\quad u = (1, \sqrt{3})$

$e = u \cdot \frac{1}{|u|}$ $\qquad |u| = \sqrt{1+3} = 2$

$e = \frac{1}{2}(1, \sqrt{3})$

parc. der. spojité na $R^2 \setminus (0,0)$ $\rightsquigarrow$ i na okolí $(1,2)$

$$\Rightarrow \frac{\partial f}{\partial e}(1,2) = \operatorname{grad} f(1,2) \cdot e$$

$$= \left(\frac{6}{25}, \frac{-3}{25}\right) \cdot \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

$$= \frac{6}{25} \cdot \frac{1}{2} - \frac{3}{25} \cdot \frac{\sqrt{3}}{2} = \frac{6 - 3\sqrt{3}}{50}$$

c) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$ $\qquad f(x,y) = \dfrac{xy}{x^2+y^2}$

$\varphi_1(t) = (t,t) \qquad \varphi_1(t) \xrightarrow{t\to 0} (0,0)$

$\lim\limits_{t\to 0} f(\varphi_1(t)) = \lim\limits_{t\to 0} \dfrac{t^2}{2t^2} = \dfrac{1}{2}$

$\varphi_2(t) = (0,t) \qquad \varphi_2(t) \xrightarrow{t\to 0} (0,0)$

$\lim\limits_{t\to 0} f(\varphi_2(t)) = \lim\limits_{t\to 0} \dfrac{0 \cdot t}{0^2+t^2} = 0$

$\frac{1}{2} \neq 0$

limita neexistuje.

Príklad 2. [20] Je daná funkcia

$$f(x, y) = 8x^3 + y^3 - 24xy + 3.$$

Nájdite jej lokálne extrémy.
Napíšte celý postup riešenia.

$D_f = R^2$

$f'_x = 24x^2 - 24y \qquad f'_y = 3y^2 - 24x$

stacionárne body:

$24x^2 - 24y = 0 \quad \rightarrow \quad x^2 = y$

$3y^2 - 24x = 0 \quad \rightarrow \quad y^2 - 8x = 0$

$x^4 = y^2 \qquad x^4 - 8x = 0$

$x(x^3 - 8) = 0$

$x(x-2)(x^2+2x+4) = 0$

riešenia $\quad x = 0 \quad x = 2$

$x^2+2x+4 = (x+2x+1) + 3 = (x+1)^2 + 3 > 0$

stac. body $\quad (0,0) \;;\; (2,4)$

$f'_{xx} = 48x \qquad f'_{xy} = -24 \qquad f'_{yx} = -24 \qquad f''_{yy} = 6y$

$$M = \begin{pmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{pmatrix}$$

$M_{(0,0)} = \begin{pmatrix} 0 & -24 \\ -24 & 0 \end{pmatrix} \qquad \det M_{(0,0)} = 0^2 - 24^2 < 0 \;\dots$

bod $(0,0)$ nie je lok. extrém

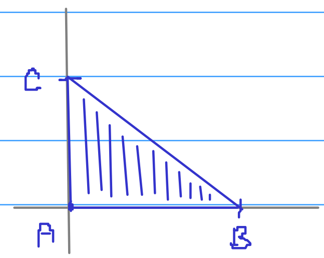
$M_{(2,4)} = \begin{pmatrix} 96 & -24 \\ -24 & 24 \end{pmatrix} \qquad \left.\begin{array}{l} \det M = 96 \cdot 24 - 24 \cdot 24 > 0 \\ 96 > 0 \end{array}\right\} \text{OLMIN}$

f má v bode $(2,4)$ OLMIN a nadobúda hodnotu -61

Pr 3. $\displaystyle\iint_M \frac{1}{y+1}\,dx\,dy$

M je trojuholník $A=[0,0]$ $B=[1,0]$ $C=[0,1]$

Riešenie: Obrázok:



Popis oblasti: $0 \le x \le 1$

M_{xy} $0 \le y \le 1-x$

(Prípadne M_{yx})

Výpočet $\displaystyle\iint_M \frac{1}{y+1}\,dx\,dy = \int_0^1 \int_0^{1-x} \frac{1}{y+1}\,dy\,dx = \int_0^1 \left[\ln|y+1|\right]_0^{1-x} dx =$

$$= \int_0^1 \ln(2-x) - 0\,dx =$$

$f' = 1$ $f = x$

$g = \ln(2-x)$ $g' = \frac{1}{2-x}\cdot(-1)$

$$= \left[x \ln(2-x)\right]_0^1 + \int_0^1 \frac{x}{2-x}\,dx = 0 - 0 + \int_0^1 -1 + \frac{2}{2-x}\,dx =$$

$$= \left[-x\right]_0^1 + \left[2\cdot(-1)\cdot\ln(2-x)\right]_0^1 = -1 - 2\cdot\left(\ln 1 - \ln 2\right) = \underline{\underline{2\ln 2 - 1}}$$

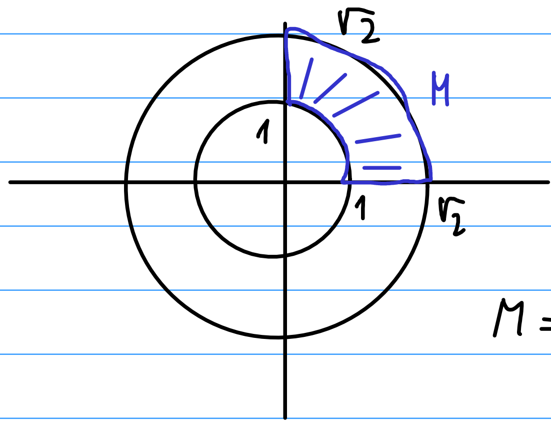
Príklad 4. [15]

Použitím substitúcie vypočítajte

$$\iint_M \frac{x}{x^2+y^2}\,dxdy,$$

ak množina M je daná nerovnosťami $1 \le x^2+y^2 \le 2$, $x \ge 0$, $y \ge 0$.

Nakreslite množinu M.
Popíšte M ako elementárnu oblasť v polárnych súradniciach.
Pri výpočte integrálu napíšte celý postup.



$x = r\cos\varphi = \phi_1(r,\varphi)$

$y = r\sin\varphi = \phi_2(r,\varphi)$

$M = M_{r\varphi}$ $\quad 1 \le r \le \sqrt{2}$

$\quad 0 \le \varphi \le \frac{\pi}{2}$

$$J(r,\varphi) = \begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix}$$

$$|J(r,\varphi)| = r\cos^2\varphi - (-r\sin^2\varphi) = r$$

$$\iint_M \frac{x}{x^2+y^2}\,dxdy = \int_1^{\sqrt{2}}\int_0^{\frac{\pi}{2}} \frac{r\cos\varphi}{r^2}\cdot r \; d\varphi\, dr$$

$$= \int_1^{\sqrt{2}} \int_0^{\frac{\pi}{2}} \cos\varphi \, d\varphi\, dr = \int_1^{\sqrt{2}} \left[\sin\varphi\right]_0^{\frac{\pi}{2}} dr =$$

$$= \int_1^{\sqrt{2}} 1 \, dr = \sqrt{2} - 1$$