RT 2019

1 Príklad $\quad f(x) = \dfrac{x^2+3x}{(x+1)^2}$ $\quad$ Monotónnosť a lokálne extrémy

$D_f = R - \{-1\}$

$$f'(x) = \frac{(2x+3)\cdot(x+1)^{\cancel{2}\,1} - (x^2+3x)\cdot 2\cancel{(x+1)}}{(x+1)^{\cancel{4}\,3}} = \frac{(2x+3)(x+1) - 2x^2 - 6x}{(x+1)^3} = \frac{\cancel{2x^2}+3x+2x+3-\cancel{2x^2}-6x}{(x+1)^3} = \frac{-x+3}{(x+1)^3}$$

Stac. body: $\quad f'(x)=0 \iff -x+3=0 \iff x=3 \Rightarrow x_1 = 3$ $\quad$ St. bod

$f'(4) = \dfrac{-1}{5^3} < 0$

$f'(0) = \dfrac{3}{1} > 0$

$f'(-2) = \dfrac{5}{(-1)^3} < 0$

	$(-\infty, -1)$	-1	$(-1, 3)$	3	$(3, \infty)$
f'	$-$	o	$+$		$-$
f	$\searrow$		$\nearrow$		$\searrow$

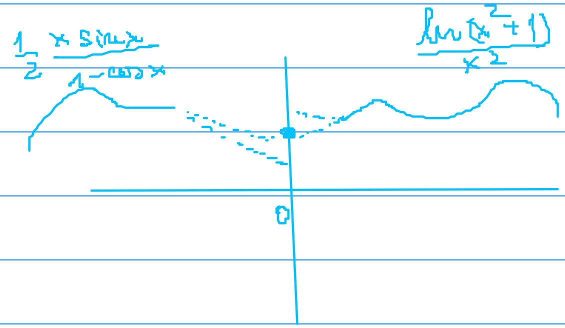
f je klesajúca na $(-\infty, -1)$ a na $\langle 3, \infty)$.
f je rastúca na $(-1, 3\rangle$.

Bod $x_1 = 3$ je bod OLMAX $\quad f(3) = \dfrac{18}{16} = \dfrac{9}{8}$

Príklad 2.

$$f(x) = \begin{cases} \dfrac{\ln(x^2+1)}{x^2} & \text{ak } x > 0 \\ 1 & \text{ak } x = 0 \\ \dfrac{1}{2}\,\dfrac{x\cdot\sin x}{1-\cos x} & \text{ak } x < 0 \end{cases}$$

Predstava:



Zistite, či je f spojitá v bode 0

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \frac{\ln(x^2+1)}{x^2} \overset{L'H}{=} \lim_{x\to 0+} \frac{\frac{1}{x^2+1}\cdot \cancel{2x}}{\cancel{2x}} = 1$$

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} \frac{1}{2}\,\frac{x\cdot\sin x}{1-\cos x} \overset{L'H}{=} \frac{1}{2}\lim_{x\to 0-} \frac{\sin x + x\cos x}{\sin x} \overset{L'H}{=} \frac{1}{2}\lim_{x\to 0-} \frac{\cos x+\cos x + x(-\sin x)}{\cos x} = \frac{1}{2}\cdot\frac{2}{1} = 1$$

Pretože $\lim_{x\to 0+} f(x) = \lim_{x\to 0-} f(x) = f(0) = 1$, tak f je spojitá v bode 0.

Príklad 3. Zistite, či konverguje alebo diverguje rad

$$\sum_{n=1}^{\infty} \frac{(n!)^2 \cdot 5^n}{(2n)!}$$

$$(2(n+1))! = (2n+2)! = (2n)! \cdot (2n+1)(2n+2)$$

D'Alembertovo kritérium:

$$\lim_{n\to\infty} \frac{\frac{(n+1)!\,(n+1)! \cdot 5^{n+1}}{(2(n+1))!}}{\frac{n! \cdot n! \cdot 5^n}{(2n)!}} = \lim_{n\to\infty} \frac{\cancel{n!}(n+1)\,\cancel{n!}(n+1) \cdot \cancel{5^n} \cdot 5 \cdot \cancel{(2n)!}}{\cancel{n!}\ \cancel{n!}\ \cancel{5^n}\ \cancel{(2n)!}\,(2n+1)\cdot(2n+2)} = \lim_{n\to\infty} \frac{(n^2+2n+1)\cdot 5}{4n^2+6n+2} =$$

$$= \frac{5}{4} > 1 \implies \text{rad diverguje}$$

Príklad 3b Zistite, kedy konverguje a kedy diverguje rad

$$\sum_{n=1}^{\infty} \frac{(n!)^2 \cdot k^n}{(2n)!}$$

k je reálna konštanta, $k > 0$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{k}{4}$$

ak $0 < k < 4$ rad konverguje

ak $4 < k$ rad diverguje

ak $k = 4$ nevieme rozhodnúť podľa D'Alemb. krit.

Príklad 4. $\displaystyle\int \frac{1}{x+2\sqrt{x}+5}\,dx =$

~~Subst.~~ $\cancel{y = x^2}$

$\cancel{dy = 2x\,dx}$

$\cancel{= \int \frac{1}{\quad}}$

Subst.: $x = t^2$ $\qquad\qquad t=\sqrt{x}$

$dx = 2t\,dt$

$$= \int \frac{2t}{t^2+2t+5}\,dt =$$

(Deliť sa nedá)

Pýtame sa: Má menovateľ korene? $t^2+2t+5=0$

$D = 4-4\cdot1\cdot5 = -16 < 0$ Nemá reálne korene.

$$= \int \frac{2t+2}{t^2+2t+5} - \frac{2}{t^2+2t+5}\,dt = \ln|t^2+2t+5| - \int\frac{2}{t^2+2t+5}\,dt =$$

$$\frac{2}{t^2+2t+5} = \frac{2}{(t^2+2t+1)+4} = \frac{2}{(t+1)^2+2^2}$$

$(t+a)^2 = t^2+2at+a^2$

$$= \ln|t^2+2t+5| - \int \frac{2}{(t+1)^2+2^2}\,dt = \ln|t^2+2t+5| - 2\int \frac{1}{s^2+2^2}\,ds = \ln|t^2+2t+5| - 2\cdot\frac{1}{2}\cdot \operatorname{arctg}\frac{s}{2} =$$

$s = t+1$

$ds = dt$

$$\int \frac{1}{x^2+1}\,dx = \operatorname{arctg} x$$

$$\boxed{\int \frac{1}{x^2+b^2}\,dx = \frac{1}{b}\operatorname{arctg}\frac{x}{b}}$$

$$\int \frac{1}{s^2+2^2}\,ds = \frac{1}{2^2}\int \frac{1}{\left(\frac{s}{2}\right)^2+1}\,ds = \frac{1}{2^{\cancel{2}}}\cdot \int \frac{1}{v^2+1}\,\cancel{2}\,dv = \frac{1}{2}\operatorname{arctg} v = \frac{1}{2}\operatorname{arctg}\frac{s}{2}$$

$s = 2v$

$ds = 2dv$

$$= \ln|t^2+2t+5| - \operatorname{arctg}\frac{t+1}{2} = \ln|x+2\sqrt{x}+5| - \operatorname{arctg}\frac{\sqrt{x}+1}{2} + c$$

Tréning:

$t^2+4t+8 = (t^2+4t+4)+4 = (t+2)^2+4$

$t^2+4t+8 = t^2+2\cdot 2t+8 = (t+2)^2+4$

$t^2-4t+8 = t^2-2\cdot 2t+4+4 = (t-2)^2+4$

$t^2+7t+15 = t^2+2\cdot\frac{7}{2}t+\frac{49}{4}+\frac{11}{4} = \left(t+\frac{7}{2}\right)^2+\frac{11}{4}$

$a^2+2\,b\cdot a+b^2$

$15 = \frac{60}{4}$

$t^2+6t+10 = t^2+2\cdot 3t+10 =$

$= (t+3)^2+1$

$$\int \frac{1}{t^2+7t+15}\,dt = \int \frac{1}{\left(t+\frac{7}{2}\right)^2+\left(\frac{\sqrt{11}}{2}\right)^2}\,dt =$$

$$= \frac{2}{\sqrt{11}}\operatorname{arctg}\frac{\left(t+\frac{7}{2}\right)\cdot 2}{\sqrt{11}}$$

Příklad 5. $\displaystyle\int_1^{e^2} \sqrt{x}\,\ln^2 x\,dx =$

Per partes $\quad f' = x^{\frac{1}{2}} \quad f = \frac{2x^{\frac{3}{2}}}{3}$

$\quad g = \ln^2 x \quad g' = 2\ln x \cdot \frac{1}{x}$

$$= \left[\frac{2}{3}x^{\frac{3}{2}} \cdot \ln^2 x\right]_1^{e^2} - \int_1^{e^2} \frac{2}{3}x^{\frac{3}{2}} \cdot 2\ln x \cdot \frac{1}{x}\,dx =$$

$$= \frac{2}{3}e^{2\cdot\frac{3}{2}} \cdot \underbrace{\ln^2(e^2)}_{2^2} - \frac{2}{3} \cdot 1 \cdot \underbrace{\ln^2 1}_{=0} - \int_1^{e^2} \frac{4}{3}x^{\frac{1}{2}} \cdot \ln x\,dx =$$

$$= \frac{8}{3}e^3 - \frac{4}{3}\int_1^{e^2} x^{\frac{1}{2}} \cdot \ln x\,dx =$$

$\quad f' = x^{\frac{1}{2}} \quad f = \frac{2}{3}x^{\frac{3}{2}}$

$\quad g = \ln x \quad g' = \frac{1}{x}$

$$= \frac{8}{3}e^3 - \frac{4}{3}\left(\left[\frac{2}{3}x^{\frac{3}{2}} \cdot \ln x\right]_1^{e^2} - \int_1^{e^2} \frac{2}{3}x^{\frac{3}{2}} \cdot \frac{1}{x}\,dx\right) = \frac{8}{3}e^3 - \frac{8}{9}\left(e^3 \cdot \ln(e^2) - 0\right) + \frac{8}{9}\int_1^{e^2} x^{\frac{1}{2}}\,dx =$$

$$= \frac{8}{3}e^3 - \frac{8}{9}e^3 \cdot 2 + \frac{8}{9}\left[\frac{2}{3}x^{\frac{3}{2}}\right]_1^{e^2} = \frac{8}{3}e^3 - \frac{16}{9}e^3 + \frac{16}{27}\left(e^3 - 1\right)$$

$n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$

$7! = \underbrace{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \cdot 7$

$7! = 6! \cdot 7$

$7! = 5! \cdot 6 \cdot 7$

$7! = 4! \cdot 5 \cdot 6 \cdot 7$

$$\frac{(n+3)!}{(n-1)!} = \frac{\cancel{(n-1)!} \cdot n \cdot (n+1)(n+2)(n+3)}{\cancel{(n-1)!}}$$