Príklad 1 $\sum_{n=2}^{\infty} \frac{3}{n^2+n-2}$ Súčet

$\lim_{n\to\infty} S_n$

$$a_n = \frac{3}{n^2+n-2} = \frac{A}{n-1} + \frac{B}{n+2} = \frac{1}{n-1} - \frac{1}{n+2}$$

$3 = A(n+2) + B(n-1)$

$3 = A\cdot 3 \qquad A=1$

$3 = B\cdot(-3) \qquad B=-1$

$n_1 = 1 \qquad n_2 = -2$

$$S_n = \frac{1}{1} - \cancel{\frac{1}{4}} + \frac{1}{2} - \cancel{\frac{1}{5}} + \frac{1}{3} - \cancel{\frac{1}{6}} + \cancel{\frac{1}{4}} - \cancel{\frac{1}{7}} + \cancel{\frac{1}{5}} - \cancel{\frac{1}{8}} + \ldots + \cancel{\frac{1}{n-3}} - \frac{1}{n} + \cancel{\frac{1}{n-2}} - \frac{1}{n+1} + \cancel{\frac{1}{n-1}} - \frac{1}{n+2}.$$

$$= 1 + \frac{1}{2} + \frac{1}{3} - \frac{1}{n} - \frac{1}{n+1} - \frac{1}{n+2}$$

$$S = \lim_{n\to\infty} S_n = 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}$$

Geometrický rad

Pr2 $\sum_{n=0}^{\infty} 2\cdot\left(\frac{3}{5}\right)^n = 2\cdot\frac{1}{1-\frac{3}{5}} = 2\,\frac{5}{2} = 5$

$\sum_{n=0}^{\infty} a\cdot q^n = a\cdot\frac{1}{1-q}$ pre $|q|<1$.

$a=2 \qquad q=\frac{3}{5} \qquad |q| = \frac{3}{5} < 1$

Pr 3. $\sum_{n=7}^{\infty} 2\left(\frac{3}{5}\right)^n = 2\left(\frac{3}{5}\right)^7 + 2\left(\frac{3}{5}\right)^8 + \ldots + \ldots = \underbrace{2\left(\frac{3}{5}\right)^7}_{a}\left[1 + \underbrace{\frac{3}{5}}_{q} + \left(\frac{3}{5}\right)^2 + \ldots\right] = 2\left(\frac{3}{5}\right)^7\cdot\frac{1}{1-\frac{3}{5}} = 5\cdot\left(\frac{3}{5}\right)^7$

Pr 4. $$\sum_{n=2}^{\infty} 5\cdot\left(-\frac{1}{3}\right)^n = \underbrace{5\left(-\frac{1}{3}\right)^2}_{a}\cdot\frac{1}{1-\left(-\frac{1}{3}\right)} = 5\cdot\frac{1}{9}\cdot\frac{3}{4} = \frac{5}{12}$$

$$|q| = \frac{1}{3} < 1$$

$$3\cdot 3^3 = 3\cdot 27 = 81$$

Pr 5. $$\sum_{n=3}^{\infty}\frac{3^{n+1}-(-4)^n}{2^{3n}} = \sum_{n=3}^{\infty}\frac{3\cdot 3^n}{8^n} - \sum_{n=3}^{\infty}\frac{(-4)^n}{8^n} = \frac{3\cdot 3^3}{8^3}\cdot\frac{1}{1-\frac{3}{8}} - \frac{(-4)^3}{8^3}\cdot\frac{1}{1-\frac{-4}{8}} = \underline{\underline{\frac{81}{512}\cdot\frac{8}{5}+\frac{1}{8}\cdot\frac{2}{3}}}$$

Konvergencia a Divergencia.

Porovnávacie Kritérium

$\sum a_n \qquad a_n \geq 0$

Pr 6 $$\sum_{n=0}^{\infty}\frac{n+1}{n^2+1}$$

$$\frac{n+1}{n^2+1} > \frac{n}{n^2+n^2} = \frac{n}{2n^2} = \frac{1}{2}\,\frac{1}{n}$$

Rad $\sum_{n=1}^{\infty}\frac{1}{n}$ je divergentný

je divergentný $\quad\Leftarrow\quad \sum_{n=1}^{\infty}\frac{n+1}{n^2+1}$ je divergentný

aj $\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}$ —||—

Pr 7 $$\sum_{n=0}^{\infty}\frac{n+1}{n^3+1}$$

$$\frac{n+1}{n^3+1} < \frac{n+n}{n^3} = \frac{2n}{n^3} = \frac{2}{n^2}$$

Rad $2\cdot\sum_{n=1}^{\infty}\frac{1}{n^2}$ je konvergentný

$$\sum_{n=0}^{\infty} \frac{n+1}{n^3+1} \text{ je konverg.} \Leftarrow \sum_{n=1}^{\infty} \frac{n+1}{n^3+1} \text{ je konvergentný} \Leftarrow$$

$$\sum_{n=n_0}^{\infty} \frac{1}{n^k} \text{ je } \begin{cases} \text{konverg. ak } k>1 \\ \text{diverg. ak } 0<k\leq 1 \end{cases}$$

Pr 8 $\displaystyle\sum_{n=0}^{\infty} \frac{n-1}{n^2-2n+5}$ Rozdiel exp. je 1 => rad diverguje ?

diverguje

$$\frac{n-1}{n^2-2n+5} > \frac{n-\frac{n}{2}}{n^2-0+5n^2} \quad \text{pre } n\geq 2$$

$$= \frac{\frac{n}{2}}{6n^2} = \frac{1}{12}\cdot\frac{1}{n} \qquad \frac{1}{12}\sum_{n=2}^{\infty}\frac{1}{n} \text{ - diverguje}$$

$$\Downarrow$$

$\sum a_n \qquad \sum b_n \qquad \displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = L \qquad 0<L<+\infty$ tak sa obe rady

chovajú rovnako

Pr 8 iný postup

$$\lim_{n\to\infty} \frac{\frac{n-1}{n^2-2n+5}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2-n}{n^2-2n+5} = \frac{1}{1} = 1$$

$$\left.\begin{array}{l} 0<1<\infty \Rightarrow \\ \sum \frac{1}{n} \text{ diverguje} \end{array}\right\} \Rightarrow \text{Pôvodný diverguje.}$$

Pr 9. $\sum_{n=2}^{\infty} \frac{n-1}{n^3-3n+7}$ $\quad \frac{1}{3} \quad 3-1=2 \quad \frac{1}{n^2} \quad 2-0=2$

$$\lim_{n\to\infty} \frac{\frac{n-1}{n^3-3n+7}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^3-n^2}{n^3-3n+7} = \frac{1}{1} = 1$$

$\sum \frac{1}{n^2}$ je konvergentný $\Rightarrow$ pôvodný je konvergentný.

Pr 10. $\sum_{n=1}^{\infty} \sin \frac{1}{n^2}$

konverguje $\Uparrow$ $\quad \lim_{n\to\infty} \frac{\sin \frac{1}{n^2}}{\frac{1}{n^2}} = \lim_{x\to 0} \frac{\sin x}{x} = 1 \Rightarrow \sum_{n=1}^{\infty} \sin \frac{1}{n^2}$ sa chovajú rovnako $\sum_{n=1}^{\infty} \frac{1}{n^2}$

$\wedge \sum_{n=1}^{\infty} \frac{1}{n^2}$ konverguje $\Rightarrow$

Pr 11 $\sum_{n=1}^{\infty} \ln \frac{1}{1-\frac{1}{n^2}}$

$$\lim_{n\to\infty} \frac{\ln \frac{1}{1-\frac{1}{n^2}}}{\frac{1}{n^2}} = \lim_{x\to 0} \frac{\ln \frac{1}{1-x}}{x} \overset{LH}{=} \lim_{x\to 0} \frac{\frac{1}{\frac{1}{1-x}} \cdot -1 \cdot (1-x)^{-2}(-1)}{1} = \lim_{x\to 0} \frac{\cancel{(1-x)} \cdot \frac{1}{(1-x)^{\cancel{2}}}}{1} = 1$$

$=$

Rad $\sum \frac{1}{n^2}$ konverguje $\Rightarrow \sum_{n=1}^{\infty} \ln \frac{1}{1-\frac{1}{n^2}}$ konverguje.

Bonus. $\sum_{n=1}^{\infty} \left(e^{\frac{1}{n}} - 1\right)$

Cachyho kritérium.

$\sum_{n=n_0}^{\infty} a_n \qquad a_n > 0 \qquad \lim_{n\to\infty} \sqrt[n]{a_n} = \ell$

$0 \le \ell < 1$ rad konv.

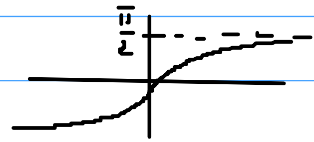
$\ell > 1$ rad diverg.

$\ell = 1$ nevieme pomocou Cauchyho

Pr 12. $\sum_{n=1}^{\infty} \left(\frac{2n}{3n+1}\right)^{2n-3}$

$$\lim_{n\to\infty} \underbrace{\left(\frac{2n}{3n+1}\right)}_{\to \frac{2}{3}}^{(2n-3)\cdot\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{2}{3+\underbrace{\frac{1}{n}}_{\to 0}}\right)^{2-\overbrace{\frac{3}{n}}^{\to 0}} = \left(\frac{2}{3}\right)^2 = \frac{4}{9} < 1 \Rightarrow \text{rad konverguje}$$

Pr 13. $\sum_{n=1}^{\infty} \left(\frac{\operatorname{arctg} n}{2}\right)^n$



$$\lim_{n\to\infty} \left(\frac{\operatorname{arctg} n}{2}\right)^{n\cdot\frac{1}{n}} = \lim_{n\to\infty} \frac{\operatorname{arctg} n}{2} = \frac{\frac{\pi}{2}}{2} = \frac{\pi}{4} < 1 \quad \text{rad konverguje}$$

Pr 14.

$$\sum_{n=1}^{\infty} \frac{3^n+1}{2^{2n+1}-7} \qquad \frac{3^n+1}{2^{2n+1}-7} < \frac{3^n}{2^{2n+1}} \qquad \sum_{n=1}^{\infty} \frac{3^n}{2^{2n+1}}$$

$$\lim_{n\to\infty} \left(\frac{3^n}{2^{2n+1}}\right)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{3}{2^{2+\frac{1}{n}}} = \frac{3}{4} < 1$$ konverguje

Pr 15.

$$\sum_{n=1}^{\infty} \left(\frac{n}{n+2}\right)^{n\cdot(n+1)}$$

$$\lim_{n\to\infty} \left(\frac{n}{n+2}\right)^{\cancel{n}\cdot(n+1)\cdot\cancel{\frac{1}{n}}}$$

$$\lim_{x\to\infty} \left(\frac{x}{x+2}\right)^{x+1} = e^{-2} = \frac{1}{e^2} < 1 \Rightarrow$$ konverguje

$$a = e^{\ln a}$$

$$\lim_{x\to\infty} \left(e^{\ln\frac{x}{x+2}}\right)^{x+1} = \lim_{x\to\infty} e^{(x+1)\cdot\ln\frac{x}{x+2}} = e^{-2}$$

$$\lim_{x\to\infty} (x+1)\cdot\ln\frac{x}{x+2} = \lim_{x\to\infty} \frac{\ln\frac{x}{x+2}}{\frac{1}{x+1}} \overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{\cancel{\frac{x}{x+2}}}\cdot\frac{\cancel{x}+2-\cancel{x}}{(x+2)^{\cancel{2}}}}{-\frac{1}{(x+1)^2}} = \lim_{x\to\infty} -\frac{2(x+1)^2}{x(x+2)} = -2$$

$$\text{"}\frac{0}{0}\text{"}$$

$$(x+1)^2 = x^2+2x+1, \qquad x(x+2) = x^2+2x$$