Cvičenie Integrovanie racionálnych funkcií

1 $\int \frac{1}{3x+2}\,dx = \frac{1}{3}\ln|3x+2| + c$

2. $\int \frac{1}{(3x+2)^4}\,dx = \int (3x+2)^{-4}\,dx = \frac{1}{3}\,\frac{(3x+2)^{-3}}{-3} = -\frac{1}{9}(3x+2)^{-3} + c$

3. $\int \frac{1}{x^2-4x+13}\,dx = \int \frac{1}{(x-2)^2+3^2}\,dx = \frac{1}{3}\operatorname{arctg}\frac{x-2}{3} + c$

$\int \frac{1}{y^2+a^2}\,da = \frac{1}{a}\operatorname{arctg}\frac{y}{a}$

$x^2-4x+13=0$

$D = 16 - 4\cdot 13 = 16-52 < 0$ Nemá reálne korene.

$(x^2-4x+4) + 9 = (x-2)^2 + 3^2$

„Doplnenie na úplný štvorec"

4. $\int \frac{1}{x^2-4x+3}\,dx \overset{*}{=}$

$x^2-4x+3=0$ $\qquad x_{1,2} = \frac{4 \pm \sqrt{16-4\cdot 1\cdot 3}}{2} = 2 \pm 1$ $\quad x_1 = 3,\ x_2 = 1$

$= (x-3)(x-1)$

$\frac{1}{x^2-4x+3} = \frac{A}{x-3} + \frac{B}{x-1}$ $\qquad 1 = A(x-1) + B(x-3)$

$x_1 = 3 \quad 1 = A\cdot 2 \qquad A = \frac{1}{2}$

$x_2 = 1 \quad 1 = B\cdot(-2) \qquad B = -\frac{1}{2}$

$\overset{*}{=} \int \frac{1/2}{x-3} - \frac{1/2}{x-1}\,dx = \frac{1}{2}\ln|x-3| - \frac{1}{2}\ln|x-1| + c$

$\int \frac{1}{2x-6}\,dx = \frac{1}{2}\ln|2x-6| =$

$= \frac{1}{2}\ln|2(x-3)| =$

$= \frac{1}{2}(\ln 2 + \ln|x-3|) =$

$$= \frac{1}{2}\ln|x-3| +$$

5. $\int \frac{x^4}{x^3+3x^2-4}\,dx = \int x-3\,dx + \int \frac{9x^2+4x-12}{x^3+3x^2-4}\,dx \overset{*}{=}$

1. krok Delenie $x^4 : (x^3+3x^2-4) = x-3 + \frac{9x^2+4x-12}{x^3+3x^2-4}$

$-(x^4+3x^3-4x)$

$-3x^3+4x$

$-(-3x^3-9x^2+12)$

$9x^2+4x-12$

2 krok Rozklad $x^3+3x^2-4=0$

	1	3	0	-4
1	1	4	4	0

1 - koreň

$(x-1)\cdot(x^2+4x+4) = (x-1)(x+2)^2$

$\overset{*}{=} \frac{x^2}{2} - 3x + \int \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}\,dx$

$\frac{}{(x-1)(x+2)^k} = \frac{A}{x-1} + \frac{B}{x+2} + \dots \frac{}{(x+2)^k}$

$9x^2+4x-12 = A(x+2)^2 + B(x-1)(x+2) + C(x-1)$

$x=1$ $1 = A9$ $A=\frac{1}{9}$

$x=-2$ $36-8-12 = C(-3)$ $C=-\frac{16}{3}$

$16 = C(-3)$

$x=0$ $-12 = 4A-2B-C = \frac{4}{9} - 2B + \frac{16}{3}$ $2B = 12 + \frac{4}{9} + \frac{16}{3}$ $B = 6 + \frac{2}{9} + \frac{8}{3} = \frac{54+2+24}{9} = \frac{80}{9}$

$$= \frac{x^2}{2} - 3x + \frac{1}{9}\int \frac{1}{x-1}\,dx + \frac{80}{9}\int \frac{1}{x+2}\,dx - \frac{16}{9}\int \frac{1}{(x+2)^2}\,dx =$$

$$= \frac{x^2}{2} - 3x + \frac{1}{9}\ln|x-1| + \frac{80}{9}\ln|x+2| + \frac{16}{9}\cdot(x+2)^{-1} + C$$

$$\underbrace{\frac{1}{x-1} + \frac{2}{x+3} + \frac{3}{(x+3)^2}}_{} = \frac{1}{x-1} + \frac{2(x+3)+3}{(x+3)^2} = \frac{1}{x-1} + \frac{2x+9}{(x+3)^2} = \frac{x^2+6x+9+2x^2+9x-2x-9}{(x-1)(x+3)^2}$$

$$= \frac{3x^2+13x}{(x-1)(x+3)^2}$$

Určitý integrál

$$\int_a^b f(x)\,dx = \left[F(x)\right]_a^b = F(b) - F(a)$$

1. $$\int_{-1}^{1} e^x\,dx = \left[e^x\right]_{-1}^{1} = e^1 - e^{-1} = e - \frac{1}{e}$$

2. $$\int_4^{20} \frac{1}{\sqrt{x+5} - \sqrt{x-4}}\,dx = \int_4^{20} \frac{\sqrt{x+5} + \sqrt{x-4}}{9}\,dx = \frac{1}{9}\int_4^{20} (x+5)^{\frac{1}{2}} + (x-4)^{\frac{1}{2}}\,dx = \frac{1}{9}\left[\frac{(x+5)^{\frac{3}{2}}}{\frac{3}{2}} + \frac{(x-4)^{\frac{3}{2}}}{\frac{3}{2}}\right]_4^{20} \overset{*}{=}$$

$$\frac{1}{\sqrt{x+5} - \sqrt{x-4}} \cdot \frac{\sqrt{x+5} + \sqrt{x-4}}{\sqrt{x+5} + \sqrt{x-4}} = \frac{\sqrt{x+5} + \sqrt{x-4}}{x+5-(x-4)} =$$

$$\overset{*}{=} \frac{1}{9}\,\frac{2}{3}\left(25^{\frac{3}{2}} - 9^{\frac{3}{2}} + 16^{\frac{3}{2}} - 0\right) = \frac{2}{27}\left(125 - 27 + 64\right) = \frac{2}{27}\,162$$

3. $$\int_1^e x \cdot \ln^2 x\,dx = \left[\frac{x^2}{2}\cdot \ln^2 x\right]_1^e - \int_1^e \frac{x^{\cancel{2}}}{\cancel{2}} \cdot \cancel{2} \ln x \cdot \frac{1}{\cancel{x}}\,dx = \frac{e^2}{2}\cdot 1 - \frac{1}{2}\cdot 0 - \int_1^e x \cdot \ln x\,dx =$$

$$f' = x \qquad f = \frac{x^2}{2}$$
$$g = \ln^2 x \qquad g' = 2\cdot \ln x \cdot \frac{1}{x}$$

$$f' = x \qquad f = \frac{x^2}{2}$$
$$g = \ln x \qquad g' = \frac{1}{x}$$

$$= \frac{e^2}{2} - \left(\left[\frac{x^2}{2}\ln x\right]_1^2 - \int_1^e \frac{x^{\cancel{2}}}{2}\cdot\frac{1}{\cancel{x}}\,dx\right) = \frac{e^2}{2} - \left(2\cdot \ln 2 - 0 - \frac{1}{2}\left[\frac{x^2}{2}\right]_1^e\right) = \frac{e^2}{2} - 2\ln 2 + \frac{1}{4}\left(e^2 - 1\right) =$$

$$= \frac{3}{4}e^2 - 2\ln 2 - \frac{1}{4}$$

4. $$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{x}{\sin^2 x}\,dx = \Big[-\operatorname{cotg} x \cdot x\Big]_{\frac{\pi}{4}}^{\frac{\pi}{3}} + \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \operatorname{cotg} x\,dx = \underbrace{-\operatorname{cotg}\left(\frac{\pi}{3}\right)}_{-\frac{1}{\sqrt{3}}} \cdot \frac{\pi}{3} + \underbrace{\operatorname{cotg}\left(\frac{\pi}{4}\right)}_{1} \cdot \frac{\pi}{4} + \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos x}{\sin x}\,dx =$$

Per partes $\quad f' = \frac{1}{\sin^2 x} \qquad f = -\operatorname{cotg} x$

$\qquad\qquad\quad g = x \qquad\qquad g' = 1$

$$= -\frac{\pi}{\sqrt{3}\cdot 3} + \frac{\pi}{4} + \Big[\ln|\sin x|\Big]_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \frac{\pi}{4} - \frac{\pi}{\sqrt{3}\cdot 3} + \ln\frac{\sqrt{3}}{2} - \ln\frac{\sqrt{2}}{2}$$

5. $$\int_1^2 \frac{e^{\frac{1}{x}}}{x^2}\,dx = -\int_1^2 e^{\frac{1}{x}}\cdot\left(-\frac{1}{x^2}\right)dx = -\int_1^{\frac{1}{2}} e^y\,dy = \int_{\frac{1}{2}}^1 e^y\,dy = \Big[e^y\Big]_{\frac{1}{2}}^1 = e^1 - e^{\frac{1}{2}}$$

Subst. $\quad y = \frac{1}{x} \qquad x = 1 \longrightarrow 1 = y$

$\qquad dy = -\frac{1}{x^2}\,dx \qquad x = 2 \longrightarrow \frac{1}{2} = y$

6. $$\int_0^1 e^{\sqrt{x}}\,dx = \int_0^1 e^y \cdot 2y\,dy = 2\int_0^1 e^y \cdot y\,dy = 2\left(\Big[e^y\cdot y\Big]_0^1 - \int_0^1 e^y\,dy\right) = 2\left(e - \Big[e^y\Big]_0^1\right) =$$

$$= 2\big(e - (e-1)\big) =$$

$$= 2$$

Subst $\quad y = \sqrt{x}$

$\qquad x = y^2$

$\qquad dx = 2y\,dy$

Per partes $\quad f' = e^y \qquad f = e^y$

$\qquad\qquad\quad g = y \qquad g' = 1$

7. $$\int_1^e \frac{1+\ln x}{x}\,dx = \int_0^1 \frac{1+y}{1}\,dy = \int_0^1 1+y\,dy = \left[y+\frac{y^2}{2}\right]_0^1 = 1+\frac{1}{2} = \frac{3}{2}$$

Subst.: $y = \ln x$

$dy = \frac{1}{x}dx$

8. $$\int_1^{27} \frac{\sqrt[3]{x^2}}{4+\sqrt[3]{x^2}}\,dx =$$

Subst. $\left(x^2 = y \quad y = x^2,\ dy = 2x\,dx \right)$ $\left(y = x^{\frac{2}{3}},\ dy = \frac{2}{3}x^{-\frac{1}{3}}dx \right)$ $x = t^3$, $dx = 3t^2\,dt$

$$= \int_1^3 \frac{t^{\frac{6}{3}}}{4+t^{\frac{6}{3}}}\,3t^2 dt = 3\int_1^3 \frac{t^4}{4+t^2}\,dt =^{*}$$

$x = 1 \quad t = 1$

$x = 27 \quad t = 3$

Dělit

$$t^4 : (t^2+4) = t^2 - 4 + \frac{16}{t^2+4}$$

$-(t^4+4t^2)$

$-4t^2$

$-(-4t^2-16)$

$+16$

$=$

$$\int \frac{1}{t^2+2^2}$$

$$\overset{*}{=} 3 \cdot \int_1^3 t^2 - 4 + \frac{16}{t^2+4}\, dt = 3 \left[\frac{t^3}{3} - 4t + 16 \cdot \frac{1}{2} \operatorname{arctg} \frac{t}{2} \right]_1^3 =$$

$$= 3\left(9 - 12 + 8 \operatorname{arctg} \frac{3}{2} - \frac{1}{3} + 4 - 8 \operatorname{arctg} \frac{1}{2} \right) = 3 \left(\frac{2}{3} + 8 \operatorname{arctg} \frac{3}{2} - 8 \operatorname{arctg} \frac{1}{2} \right)$$