# Int. rac. funkcií

1. $\int \frac{x^2+1}{x^3-1}\,dx =$

$$\frac{x^2+1}{x^3-1} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1} \qquad x^2+1 = A(x^2+x+1) + (Bx+C)(x-1)$$

$$x^3-1 = (x-1)(x^2+x+1)$$

$$D = 1-4.1.1 < 0$$

$$x=1 \qquad 2 = 3A \qquad A = \frac{2}{3}$$

$$x=0 \qquad 1 = \frac{2}{3} + (-1)C \qquad C = -\frac{1}{3}$$

$$x^2: \qquad 1 = A + B \qquad B = \frac{1}{3}$$

$$= \int \frac{2/3}{x-1} + \frac{1/3\,x - 1/3}{x^2+x+1}\,dx = \frac{2}{3}\ln|x-1| + \frac{1}{3}\frac{1}{2}\int \frac{2(x-1)}{x^2+x+1}\,dx = \frac{2}{3}\ln|x-1| + \frac{1}{6}\int \frac{2x+1}{x^2+x+1} - \frac{3}{x^2+x+1}\,dx =$$

$2x+1$

$$= \frac{2}{3}\ln|x-1| + \frac{1}{6}\ln|x^2+x+1| - \frac{1}{2}\int \frac{1}{(x^2+x+\frac{1}{4})+\frac{3}{4}}\,dx = \frac{2}{3}\ln|x-1| + \frac{1}{6}\ln|x^2+x+1| - \frac{1}{2}\int \frac{1}{\underbrace{(x+\frac{1}{2})}_{y}^2 + \underbrace{\left(\frac{\sqrt{3}}{2}\right)}_{a}^2}\,dx =$$

$$a^2 + 2ab + b^2 = (a+b)^2$$

$$x^2 + 2x\frac{1}{2} + \frac{1}{4} = \left(x + \frac{1}{2}\right)^2$$

$$= \frac{2}{3}\ln|x-1| + \frac{1}{6}\ln|x^2+x+1| - \frac{1}{\cancel{2}}\,\frac{1}{\frac{\sqrt{3}}{\cancel{2}}}\,\mathrm{arctg}\,\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{2}{3}\ln|x-1| + \frac{1}{6}\ln|x^2+x+1| - \frac{1}{\sqrt{3}}\,\mathrm{arctg}\,\frac{2x+1}{\sqrt{3}} + C$$

Pr 2. $\int \frac{1}{\cos x}\,dx = \int \frac{\cos x}{\cos^2 x}\,dx = \int \frac{\cos x}{1-\sin^2 x}\,dx = \int \frac{1}{1-y^2}\,dy = \int \frac{A}{1-y} + \frac{B}{1+y}\,dy =$

Sub. $y = \sin x$

$dy = \cos x\,dx$

$1 = A(1+y) + B(1-y)$

$y=1 \quad 1 = 2A \quad A = 1/2$

$y=-1 \quad 1 = 2B \quad B = 1/2$

$= \frac{1}{2}\int \frac{1}{1-y}\,dy + \frac{1}{2}\int \frac{1}{1+y}\,dy =$

$= -\frac{1}{2} \cdot \ln|1-y| + \frac{1}{2}\ln|1+y| + c = \frac{1}{2}\ln|1+\sin x| - \frac{1}{2}\ln|1-\sin x| + C = \ln\sqrt{\frac{1+\sin x}{1-\sin x}} + C$

Určitý integrál

$$\int_a^b f(x)\,dx = \left[F(x)\right]_a^b = F(b) - F(a)$$ číslo

3. $$\int_{-1}^{1} e^x\,dx = \left[e^x\right]_{-1}^{1} = e^1 - e^{-1} = e - \frac{1}{e}$$

4. $$\int_4^{20} \frac{1}{\sqrt{x+5} - \sqrt{x-4}}\,dx = \int_4^{20} \frac{\sqrt{x+5} + \sqrt{x-4}}{x+5-(x-4)}\,dx = \frac{1}{9}\int_4^{20} \sqrt{x+5} + \sqrt{x-4}\,dx = \frac{1}{9}\left[\frac{2}{3}(x+5)^{\frac{3}{2}} + \frac{2}{3}(x-4)^{\frac{3}{2}}\right]_4^{20} =$$

$$= \frac{2}{27}\left(25^{\frac{3}{2}} + 16^{\frac{3}{2}} - 9^{\frac{3}{2}} - 0\right) = \frac{2}{27}(125 + 64 - 27) = \frac{2}{27} \cdot 162 = \frac{2}{3}\,18$$

5. $$\int_1^e x \cdot \ln^2 x\,dx = \left[\frac{x^2}{2}\ln^2 x\right]_1^e - \int_1^e \frac{x^{\cancel{2}}}{\cancel{2}} \cdot \cancel{2} \cdot \ln x \cdot \frac{1}{\cancel{x}}\,dx =$$

p.p. $f' = x$ $\quad f = \frac{x^2}{2}$

$g = \ln^2 x$ $\quad g' = 2 \cdot \ln x \cdot \frac{1}{x}$

$$= \frac{e^2}{2}\ln^2 e - \frac{1}{2}\ln^2 1 - \int_1^e x \cdot \ln x\,dx =$$

$\ln e = 1$ $\quad = 0$

$f' = x$ $\quad f = \frac{x^2}{2}$

$g = \ln x$ $\quad g' = \frac{1}{x}$

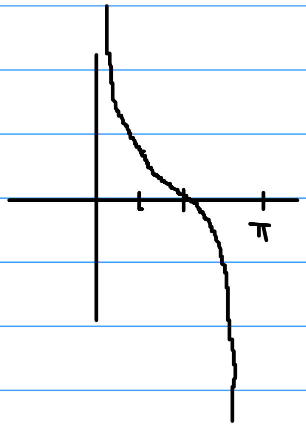
$$= \frac{e^2}{2} - \left(\left[\frac{x^2}{2} \cdot \ln x\right]_1^e - \int_1^e \frac{x^{\cancel{2}}}{2} \cdot \frac{1}{\cancel{x}}\,dx\right) = \cancel{\frac{e^2}{2}} - \left(\cancel{\frac{e^2}{2}} - 0 - \frac{1}{2}\left[\frac{x^2}{2}\right]_1^e\right) = +\frac{1}{4}\left(e^2 - 1\right)$$

6. $$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{x}{\sin^2 x}\,dx = \left[-\operatorname{ctg} x \cdot x\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} + \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \operatorname{ctg} x\,dx = -\operatorname{ctg}\frac{\pi}{3}\cdot\frac{\pi}{3} + \operatorname{ctg}\frac{\pi}{4}\cdot\frac{\pi}{4} + \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos x}{\sin x}\,dx =$$

$f' = \frac{1}{\sin^2 x}$ $\quad f = -\operatorname{ctg} x$

$g = x$ $\quad g' = 1$

$$= \frac{\pi}{4}\cdot 1 - \frac{\pi}{3}\cdot\frac{1}{\sqrt{3}} + \left[\ln|\sin x|\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} =$$




$$= \frac{\pi}{4} - \frac{\pi}{3\sqrt{3}} + \ln\left(\sin\frac{\pi}{3}\right) - \ln\left(\sin\frac{\pi}{4}\right) = \frac{\pi}{4} - \frac{\pi}{3\sqrt{3}} + \ln\frac{\frac{\sqrt{3}}{\cancel{2}}}{\frac{\sqrt{2}}{\cancel{2}}} =$$

$$= \frac{\pi}{4} - \frac{\pi}{3\sqrt{3}} + \frac{1}{2}\ln\frac{3}{2}$$

7. $$-\int_1^2 \frac{e^{\frac{1}{x}}}{-x^2}\,dx = -\int_1^{1/2} e^y\,dy = \int_{1/2}^{1} e^y\,dy = \left[e^y\right]_{1/2}^{1} = e - \sqrt{e}$$

Sub: $y = \frac{1}{x}$ $\qquad x = 1 \Rightarrow y = 1$

$dy = -\frac{1}{x^2}dx$ $\qquad x = 2 \Rightarrow y = \frac{1}{2}$

8. $$\int_1^{27} \frac{\sqrt[3]{x^2}}{4+\sqrt[3]{x^2}}\,dx = \int_1^3 \frac{t^2}{4+t^2}\cdot 3t^2\,dt = 3\int_1^3 \frac{t^4}{4+t^2}\,dt =$$

$t^4 : t^2+4 = t^2 - 4 + \frac{16}{t^2+4}$

$-(t^4+4t^2)$

$-4t^2$

$-(-4t^2-16)$

$+16$

Sub: $x = t^3$ $\quad t = \sqrt[3]{x}$ $\quad x = 1 \Rightarrow t = 1$

$dx = 3t^2\,dt$ $\qquad x = 27 \Rightarrow t = \sqrt[3]{27} = 3$

$$= 3\int_1^3 t^2 - 4 + \frac{16}{t^2+4}\,dt = 3\left[\frac{t^3}{3} - 4t + 16\cdot\frac{1}{2}\operatorname{arctg}\frac{t}{2}\right]_1^3 = 3\left(9 - 12 + 8\operatorname{arctg}\frac{3}{2} - \frac{1}{3} + 4 - 8\operatorname{arctg}\frac{1}{2}\right) =$$

$$= 3\left(\frac{2}{3} + 8\operatorname{arctg}\frac{3}{2} - 8\operatorname{arctg}\frac{1}{2}\right)$$

9. $$\int_0^1 e^{\sqrt{x}}\,dx = \int_0^1 e^t\, 2t\,dt = 2\int_0^1 t\cdot e^t\,dt =$$

Sub. $x = t^2$ $\quad t = \sqrt{x}$ $\quad x = 0 \Rightarrow t = 0$

$dx = 2t\,dt$ $\quad x = 1 \Rightarrow t = 1$

$f' = e^t \quad f = e^t$

$g = t \quad g' = 1$

$$= 2\left(\left[t\cdot e^t\right]_0^1 - \int_0^1 e^t\,dt\right) = 2\left(e - 0 - \left[e^t\right]_0^1\right) = 2(e - e + 1) = \underline{\underline{2}}$$

10. $$\int_1^e \frac{1 + \ln x}{x}\,dx = \int_0^1 \frac{1 + y}{1}\,dy = \left[y + \frac{y^2}{2}\right]_0^1 = 1 + \frac{1}{2} - 0 - 0 = \frac{3}{2}$$

Sub. $y = \ln x$ $\quad x = 1 \Rightarrow y = \ln 1 = 0$

$dy = \frac{1}{x}\,dx$ $\quad x = e \Rightarrow y = \ln e = 1$