Cvičenie 11. týždeň

Pr 1. $\int 2x^2 + \frac{1}{x} - 2\cos x \, dx = 2\,\frac{x^3}{3} + \ln|x| - 2\sin x + c$ na I takom, že $0 \notin I$

Pr 2. $\int \frac{1}{1+x^2}\, dx = \operatorname{arctg} x + c$

Pr 3 $\int \frac{1}{\sqrt{x}} + \frac{1}{\sqrt{1-x^2}}\, dx = \int x^{-\frac{1}{2}} + \frac{1}{\sqrt{1-x^2}}\, dx = 2x^{\frac{1}{2}} + \arcsin x + c$

Metóda per partes $\int \underbrace{f' \cdot g}_{\text{ťažký}}\, dx = f \cdot g - \int \underbrace{f \cdot g'}_{\text{ľahký}}\, dx$

Pr 4 $\int x \sin x \, dx = (-\cos x)\cdot x + \int \cos x \, dx = \underline{-x\cdot\cos x + \sin x + c}$

$f' = \sin x \qquad f = -\cos x$

$g = x \qquad g' = 1$

Pr 5. $\int (5x^2-7x+3)\cdot\sin x \, dx = (-\cos x)\cdot(5x^2-7x+3) + \int \cos x \cdot (10x-7)\, dx \overset{*}{=}$

$f' = \sin x \qquad f = -\cos x \qquad\qquad f' = \cos x \qquad f = \sin x$

$g = 5x^2-7x+3 \qquad g' = 10x-7 \qquad\qquad g = 10x-7 \qquad g' = 10$

$\overset{*}{=} (-5x^2+7x-3)\cos x + \left((10x-7)\sin x - \int \sin x \cdot 10\, dx\right) = (-5x^2+7x-3)\cos x + (10x-7)\sin x + 10\cos x \; (+c)$

pre $x>0$

Pr6. $\int x^2 \cdot \ln x \, dx = \frac{x^3}{3} \cdot \ln x - \int \frac{x^3}{3} \cdot \frac{1}{x} \, dx = \frac{x^3}{3} \ln x - \frac{1}{3} \int x^2 dx = \frac{x^3}{3} \cdot \ln x - \frac{1}{3} \cdot \frac{x^3}{3} + c$

$f' = x^2 \qquad f = \frac{x^3}{3}$

$g = \ln x \qquad g' = \frac{1}{x}$

Pr7. $\int \ln^2 x \, dx = x \cdot \ln^2 x - \int \cancel{x} \cdot 2 \cdot \ln x \cdot \frac{1}{\cancel{x}} \, dx = x \ln^2 x - 2 \int \ln x \, dx \overset{*}{=}$

$\ln^2 x = (\ln x)^2$

$(\ln(x^2)$ – je iná funkcia$)$

$f' = 1 \qquad f = x$

$g = \ln^2 x \qquad g' = 2 \ln x \cdot \frac{1}{x}$

$f' = 1 \qquad f = x$

$g = \ln x \qquad g' = \frac{1}{x}$

$\overset{*}{=} x \ln^2 x - 2 \left(x \cdot \ln x - \int \cancel{x} \cdot \frac{1}{\cancel{x}} dx \right) = x \ln^2 x - 2 x \cdot \ln x + 2x + c$

Pr8. $\int x^3 \operatorname{arctg} x \, dx = \frac{x^4}{4} \cdot \operatorname{arctg} x - \int \frac{x^4}{4} \cdot \frac{1}{1+x^2} \, dx = \frac{x^4}{4} \operatorname{arctg} x - \frac{1}{4} \int \frac{x^4}{1+x^2} \, dx \overset{*}{=}$

$f' = x^3 \qquad f = \frac{x^4}{4}$

$g = \operatorname{arctg} x \qquad g' = \frac{1}{1+x^2}$

Delenie:

$$\begin{aligned}
& x^4 : (x^2+1) = x^2 - 1 + \frac{1}{x^2+1} \\
& -(x^4 + x^2) \\
& \quad -x^2 \\
& -(-x^2 - 1) \\
& \quad 1
\end{aligned}$$

$$= \frac{x^4}{4} \operatorname{arctg} x - \frac{1}{4} \int x^2 - 1 + \frac{1}{x^2+1} \, dx = \frac{x^4}{4} \operatorname{arctg} x - \frac{1}{4} \left(\frac{x^3}{3} - x + \operatorname{arctg} x \right) + c$$

Pr 9.

$$\int \operatorname{arctg} x \, dx = x \operatorname{arctg} x - \int x \cdot \frac{1}{1+x^2} \, dx = x \cdot \operatorname{arctg} x - \frac{1}{2} \int \frac{2x}{1+x^2} \, dx = x \cdot \operatorname{arctg} x - \frac{1}{2} \ln(1+x^2) + c$$

$f' = 1 \qquad f = x$

$g = \operatorname{arctg} x \qquad g' = \frac{1}{1+x^2}$

$$\int \frac{\varphi'(x)}{\varphi(x)} \, dx = \ln|\varphi(x)|$$

Substituční metóda.

Pr 1. $\int \cos 3x \, dx = \int \cos y \, \frac{1}{3} \, dy = \frac{1}{3} \sin y = \frac{1}{3} \sin 3x + c$

Subst $\quad y = 3x$

$\qquad dy = 3 \, dx$

Pr 2. $\int \sqrt{7-3x} \, dx = -\frac{1}{3} \int \sqrt{7-3x} \cdot (-3) \, dx = -\frac{1}{3} \int \sqrt{y} \cdot dy = -\frac{1}{3} \, \frac{2 y^{\frac{3}{2}}}{3} = -\frac{2}{9} (7-3x)^{\frac{3}{2}} + c$

$\sqrt{y} = y^{\frac{1}{2}}$

Subst $\quad y = 7 - 3x$

$\qquad dy = -3 \cdot dx$

Pr3. $$\int \frac{x^3+5x-3}{2x-1}\,dx = \int \frac{1}{2}x^2+\frac{1}{4}x+\frac{21}{8}-\frac{3}{8}\,\frac{1}{2x-1}\,dx = \frac{1}{2}\,\frac{x^3}{3}+\frac{1}{4}\,\frac{x^2}{2}+\frac{21}{8}x-\frac{3}{8}\int\frac{1}{2x-1}\,dx \overset{*}{=}$$

Delenie $$(x^3+5x-3):(2x-1) = \frac{1}{2}x^2+\frac{1}{4}x+\frac{21}{8}-\frac{3/8}{2x-1}$$

$$-\left(x^3-\frac{1}{2}x^2\right)$$
$$\frac{1}{2}x^2+5x-3$$
$$-\left(\frac{1}{2}x^2-\frac{1}{4}x\right)$$
$$\frac{21}{4}x-3$$
$$-\left(\frac{21}{4}x-\frac{21}{8}\right)$$
$$\frac{21}{8}-\frac{24}{8}$$

Subst. $y=2x-1$

$dy = 2\,dx$

$$\overset{*}{=} \frac{x^3}{6}+\frac{x^2}{8}+\frac{21}{8}x-\frac{3}{8}\cdot\frac{1}{2}\int\frac{1}{y}\,dy =$$

$$= \frac{x^3}{6}+\frac{x^2}{8}+\frac{21}{8}x-\frac{3}{16}\ln|2x-1|+C$$

pre $x>\frac{1}{2}$

alebo $x<\frac{1}{2}$

Pr4. $$\int \sin^4 x\,\cos x\,dx = \int y^4\cdot dy = \frac{y^5}{5} = \frac{\sin^5 x}{5} = \frac{(\sin x)^5}{5}+C$$

Subst. $y=\sin x$

$dy = \cos x\,dx$

Pr5 $$\int\frac{\cos x}{1+\sin^2 x}\,dx = \int\frac{1}{1+y^2}\,dy = \operatorname{arctg}(\sin x)+C$$

$y=\sin x$

$dy=\cos x\,dx$

Sk.:

$F(x) = -e^{-\frac{x^2}{2}} + c$

$F'(x) = \cancel{+}\, e^{-\frac{x^2}{2}} \cdot \cancel{+} \frac{\cancel{2}x}{\cancel{2}}$

Pr6. $\int x \cdot e^{-\frac{x^2}{2}} dx = -\int e^{-\frac{x^2}{2}} (-1) x dx = -\int e^{y} dy = -e^{y} = -e^{-\frac{x^2}{2}} + C$ //

$y = -\frac{x^2}{2}$

$dy = -x\,dx$

Pr7. $\int x \cdot \sqrt{1+x^2}\, dx = \frac{1}{2} \int \sqrt{1+x^2}\, 2x\,dx = \frac{1}{2} \int \sqrt{y}\, dy = \frac{1}{\cancel{2}} \frac{\cancel{2} y^{\frac{3}{2}}}{3} = \frac{1}{3} (1+x^2)^{\frac{3}{2}} + C$

$y = 1+x^2$

$dy = 2x dx$

Pr8. $\int \sqrt{4-x^2}\, dx \overset{*}{=}$

~~$y = 4-x^2$~~
~~$dy = -2x dx$~~ NIE

~~$y =$~~
~~$dy =$~~ NIE

$(2-x)^2 = 4 - 4x + x^2$ Nepomohlo

- $x = 2 \cdot \sin t$ Prečo? $4 - x^2 = 4 - 4\sin^2 t = 4(1-\sin^2 t) = 4\cos^2 t$

$dx = 2\cos t\, dt$ $\sqrt{4-x^2} = \; .\;.\;.\;.\;. = \sqrt{4} \cdot \sqrt{\cos^2 t} = 2\cos t$

$\overset{*}{=} \int 2\cos t \cdot 2 \cdot \cos t\, dt = 4 \int \cos^2 t\, dt = 4 \cdot \int \frac{1+\cos 2t}{2} dt = 2\left(t + \frac{\sin 2t}{2}\right) = 2t + \sin 2t \overset{*}{=}$

$\cos^2 t = \frac{1+\cos 2t}{2}$

- $x = 2\sin t$

$\frac{x}{2} = \sin t$

$t = \arcsin\frac{x}{2}$

$\stackrel{*}{=} 2\arcsin\frac{x}{2} + \sin\left(2\arcsin\frac{x}{2}\right) + C =$

$\sin 2\alpha = 2\sin\alpha\cos\alpha$

$= 2\arcsin\frac{x}{2} + 2\sin\left(\arcsin\frac{x}{2}\right)\cdot\cos\left[\arcsin\frac{x}{2}\right] + C =$

$= 2\arcsin\frac{x}{2} + \cancel{2}\frac{x}{\cancel{2}}\cdot\sqrt{1-\left(\frac{x}{2}\right)^2} + C$

Pr 9. $\displaystyle\int\frac{\sqrt{x}}{\sqrt{x}+1}\,dx = \int\frac{\sqrt{t^2}}{\sqrt{t^2}+1}\,2t\,dt \stackrel{*}{=} \int\frac{2t^2}{t+1}\,dt = 2\int\frac{t^2}{t+1}\,dt = 2\int t-1+\frac{1}{t+1}\,dt \stackrel{*}{=}$

Subst. $x = t^2$

$dx = 2t\,dt$

pre $t>0$

$t^2 : (t+1) = t - 1 + \frac{1}{t+1}$

$-(t^2+t)$

$-t$

$-(-t-1)$

1

$\stackrel{*}{=} 2\left(\frac{t^2}{2} - t + \ln|t+1|\right) = t^2 - 2t + 2\ln(t+1) =$

$* \displaystyle\int\frac{t}{t+1}\,2t\,dt$

Ak $x = t^2$ tak $t = \sqrt{x}$

$= x - 2\sqrt{x} + 2\ln(\sqrt{x}+1) + C$

Pr 10.

$$\int \frac{\sqrt{x}-2}{x+1}\,dx = \int \frac{t-2}{t^2+1}\,2t\,dt = 2\int \frac{t^2-2t}{t^2+1}\,dt = 2\int 1-\frac{2t+1}{t^2+1}\,dt = 2t-2\int \frac{2t+1}{t^2+1}\,dt =$$

Subst $x = t^2 \Rightarrow t = \sqrt{x}$

$dx = 2t\,dt$

$$t^2-2t : (t^2+1) = 1 - \frac{2t+1}{t^2+1}$$

$$\frac{-(t^2+1)}{-2t-1}$$

$$= 2t - 2\int \frac{2t}{t^2+1} + \frac{1}{t^2+1}\,dt = 2t - 2\cdot\left(\ln(t^2+1) + \operatorname{arctg} t\right) =$$

$$= 2\sqrt{t} - 2\left(\ln(x+1) + \operatorname{arctg}\sqrt{x}\right) + C$$