Trigonometrická subst.

1. $\int \sin^4 x \underline{\cos x\, dx} = \int y^4\, dy = \frac{1}{5} y^5 = \frac{1}{5} \sin^5 x + c$

Subst. $y = \sin x$

$dy = \cos x\, dx$

2. $\int \frac{\cos x}{1+\sin^2 x}\, dx = \int \frac{1}{1+y^2}\, dy = \operatorname{arctg} y = \operatorname{arctg}(\sin x) + c$

ditto

Iné subst.

3. $\int \frac{x}{x^2+1}\, dx = \frac{1}{2}\int \frac{2x}{x^2+1}\, dx = \frac{1}{2}\int \frac{1}{y}\, dy = \frac{1}{2} \ln|y| = \frac{1}{2} \ln(x^2+1) + c = \ln\sqrt{x^2+1} + c$

$y = x^2+1$

$dy = 2x\,dx$

$\int \frac{\varphi'(x)}{\varphi(x)} = \ln|\varphi(x)|$

4. $\int \underbrace{x}\, e^{-\frac{x^2}{2}}\, dx = \int e^{-y}\, dy = -e^{-y} = -e^{-\frac{x^2}{2}}$

$y = \frac{x^2}{2}$

$dy = x\, dx$

5. $\int x\cdot\sqrt{1+x^2}\,dx = \frac{1}{2}\int\sqrt{1+x^2}\cdot 2x\,dx = \frac{1}{2}\int\sqrt{y}\,dy = \frac{1}{\cancel{2}}\,\frac{y^{\frac{3}{2}}}{\frac{3}{\cancel{2}}} = \frac{1}{3}(1+x^2)^{\frac{3}{2}} + c$

$y = 1+x^2$

$dy = 2x\,dx$

6.

$\int\sqrt{4-x^2}\,dx = \int\sqrt{4-4\sin^2 t}\cdot 2\cos t\,dt =$

$\sqrt{1-\sin^2\alpha} = \sqrt{\cos^2\alpha} = \cos\alpha$ ak $\cos\alpha \geq 0$

$x = 2\sin t$ $\qquad \frac{x}{2} = \sin t$

$dx = 2\cos t\,dt$ $\qquad t = \arcsin\frac{x}{2}$

$= \int\sqrt{4(1-\sin^2 t)}\cdot 2\cos t\,dt = \int 2^2\cos^2 t\,dt =$

$\cos^2\alpha = \frac{1+\cos 2\alpha}{2}$

$= \int 4\,\frac{1+\cos 2t}{2}\,dt = 2\int 1\,dt + 2\int\cos 2t\,dt = 2t + \cancel{2}\,\frac{\sin 2t}{\cancel{2}} = 2\arcsin\frac{x}{2} + \sin 2\left(\arcsin\frac{x}{2}\right) + c =$

$\sin 2\alpha = 2\sin\alpha\cos\alpha$

$= 2\arcsin\frac{x}{2} + 2\underbrace{\sin\left(\arcsin\frac{x}{2}\right)}_{\frac{x}{2}}\cdot\cos\left(\arcsin\frac{x}{2}\right) + c = 2\arcsin\frac{x}{2} + x\cdot\sqrt{1-\sin^2\left(\arcsin\frac{x}{2}\right)} =$

$= 2\arcsin\frac{x}{2} + x\cdot\sqrt{1-\frac{x^2}{4}} + c$

7.

$\int\sqrt{1-2x^2}\,dx = \int\sqrt{1-\cancel{2}\cdot\frac{1}{\cancel{2}}\sin^2 t}\cdot\frac{1}{\sqrt{2}}\cos t\,dt = \frac{1}{\sqrt{2}}\int\cos^2 t\,dt = \frac{1}{\sqrt{2}}\int\frac{1+\cos 2t}{2}\,dt = \frac{1}{\sqrt{2}\cdot 2}\,t + \frac{1}{\sqrt{2}\cdot 2^2}\sin 2t + c =$

$\arcsin\sqrt{2}x = t$ $\qquad x = \frac{1}{\sqrt{2}}\sin t$ $\qquad dx = \frac{1}{\sqrt{2}}\cos t\,dt$

$= \frac{1}{2\sqrt{2}}\arcsin(\sqrt{2}x) + \frac{1}{4\cdot\sqrt{2}}\sin\left(2\arcsin\sqrt{2}x\right) + c$

8.

$$\int \frac{\sqrt{x}}{\sqrt{x}-1}\,dx = \int \frac{t}{t-1}\,2t\,dt = 2\int \frac{t^2}{t-1}\,dt = 2\int t+1+\frac{1}{t-1}\,dt = 2\left(\frac{t^2}{2}+t+\ln|t-1|\right)+c =$$

$$= x+2\sqrt{x}+2\ln|\sqrt{x}-1|+c$$

$x = t^2$

$dx = 2t\,dt$

$\sqrt{x} = t$

$$\begin{array}{l} t^2 : (t-1) = t+1+\dfrac{1}{t-1} \\ -(t^2-t) \\ \hline \quad t \\ \quad -(t-1) \\ \hline \quad\quad 1 \end{array}$$

9.

$$\int \frac{\sqrt{x}-2}{x+1}\,dx = \int \frac{t-2}{t^2+1}\,2t\,dt = 2\int \frac{t^2-2t}{t^2+1}\,dt = 2\int 1-\frac{2t}{t^2+1}-\frac{1}{t^2+1}\,dt = 2\left(t-\ln(t^2+1)-\operatorname{arctg} t\right)+c =$$

dtto

$$\begin{array}{l} t^2-2t : (t^2+1) = 1-\dfrac{2t+1}{t^2+1} \\ -(t^2+1) \\ \hline \quad -2t-1 \end{array}$$

$$-\frac{2t+1}{t^2+1} = -\left(\frac{2t}{t^2+1}+\frac{1}{t^2+1}\right)$$

$$= 2\left(\sqrt{x}-\ln|(x+1)|-\operatorname{arctg}\sqrt{x}\right)+c$$

10.

$$\int \frac{1}{\sqrt[3]{x}\,(x-1)}\,dx = \int \frac{1}{\cancel{t}\,(t^3-1)}\,3t^{\cancel{2}}\,dt = 3\int \frac{t}{t^3-1}\,dt = 3\int \frac{1/3}{t-1}+\frac{-\frac{1}{3}t+\frac{1}{3}}{t^2+t+1}\,dt = *$$

$x = t^3$

$dx = 3t^2\,dt$

$\sqrt[3]{x} = t$

$$t^3-1 = (t-1)(t^2+t+1)$$

$$\frac{t}{t^3-1} = \frac{t}{(t-1)(t^2+t+1)} = \frac{A}{t-1} + \frac{Bt+C}{t^2+t+1} \qquad t = A(t^2+t+1) + (Bt+C)(t-1)$$

$t=1 \quad 1 = 3A \qquad A = \frac{1}{3}$

$t=0 \quad 0 = A + (-1)C = A - C \qquad C = \frac{1}{3}$

buď $t^2 \qquad 0 = A + B \qquad B = -\frac{1}{3}$

alebo zvolíme $\quad t=-1 \qquad -1 = A + (-B+C)(-2)$

$$-1 = \frac{1}{3} + 2B - 2\cdot\frac{1}{3}$$

$$-1 - \frac{1}{3} + \frac{2}{3} = 2B$$

$$-\frac{2}{3} = 2B$$

$$* = \int \frac{1}{t-1}\,dt + \int \frac{-t+1}{t^2+t+1}\,dt = \ln|t-1| - \int \frac{t-1}{t^2+t+1}\,dt = \ln|t-1| - \frac{1}{2}\int \frac{2t-2}{t^2+t+1}\,dt =$$

$$(t^2+t+1)' = 2t+1$$

$$= \ln|t-1| - \frac{1}{2}\int \boxed{\frac{2t+1}{t^2+t+1}} - \frac{3}{t^2+t+1}\,dt = \ln|t-1| - \frac{1}{2}\ln|t^2+t+1| + \frac{3}{2}\int \frac{1}{\underbrace{\left(t^2+t+\frac{1}{4}\right)}_{\left(t+\frac{1}{2}\right)^2}+\frac{3}{4}}\,dt =$$

$$t^2+t = t^2 + 2\frac{t}{2} + \frac{1}{4} - \frac{1}{4}$$

$$t^2+t+1 = \left(t+\frac{1}{2}\right)^2 + \frac{3}{4}$$

$$= \ln|t-1| - \frac{1}{2}\ln|t^2+t+1| + \frac{3}{2}\int \frac{1}{\left(t+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\,dt =$$

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\,\mathrm{arctg}\,\frac{x}{a}$$

$$= \ln|t-1| - \frac{1}{2}\ln|t^2+t+1| + \frac{3}{\cancel{2}}\,\mathrm{arctg}\,\frac{t+\frac{1}{2}}{\sqrt{3}/2} \cdot \frac{1}{\sqrt{3}/\cancel{2}} + c =$$

$$= \ln|\sqrt[3]{x}-1| - \frac{1}{2}\ln\left|x^{\frac{2}{3}} + x^{\frac{1}{3}} + 1\right| + \sqrt{3}\,\mathrm{arctg}\,\frac{2\sqrt[3]{x}+1}{\sqrt{3}} + C$$

Integrowanie rac. funkcji.

11. $$\frac{1}{4}\int \frac{1\cdot 4}{4x-5}\,dx = \frac{1}{4}\ln|4x-5| + C$$

12. $$\int \frac{1}{(4x-5)^3}\,dx = \frac{1}{4}\int \frac{1}{(4x-5)^3}\cdot 4\,dx = \frac{1}{4}\int \frac{1}{y^3}\,dy = \frac{1}{4}\,\frac{y^{-2}}{-2} = -\frac{1}{8}\,\frac{1}{(4x-5)^2} + C$$

$y = 4x-5$

$dy = 4\,dx$

13. $$\int \frac{1}{x^2-4x+5}\,dx = \int \frac{1}{(x-2)^2+1^2} = \frac{1}{1}\,\mathrm{arctg}\,\frac{x-2}{1} = \mathrm{arctg}(x-2) + C$$

$D = 16 - 4\cdot 1\cdot 5 = -4 < 0$

$x^2 - 4x + 4 + 1 = (x-2)^2 + 1$

$x^2 - 2\cdot x\cdot 2$

14. $$\int \frac{x^4}{x^3+3x^2-4}\,dx = \int x - 3 + \frac{9x^2+4x-12}{x^3+3x^2-4}\,dx =$$

$$x^4 : (x^3+3x^2-4) = x - 3 + \frac{9x^2+4x-12}{x^3+3x^2-4}$$

$-(x^4+3x^3-4x)$

$-3x^3+4x$

$-(-3x^3-9x^2+12)$

$9x^2+4x-12$

$$= \frac{x^2}{2} - 3x + \int \frac{9x^2+4x-12}{x^3+3x^2-4}\,dx =$$

	1	3	0	-4
1	1	4	4	0

$$(x-1)(x^2+4x+4)$$

$$x^3+3x^2-4 : (x-1) = x^2+4x+4$$

$$\underline{-x^3-x^2}$$

$$4x^2-4$$

$$\underline{-4x^2-4x}$$

$$4x-4$$

$$x^2+4x+4 = (x+2)^2$$

$$= \frac{x^2}{2} - 3x + \int \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}\,dx$$

$$9x^2+4x-12 = A(x+2)^2 + B(x-1)(x+2) + C(x-1)$$

$x=1$ $\quad 1 = 9A$ $\quad A = \frac{1}{9}$

$x=-2$ $\quad 16 = -3C$ $\quad C = -\frac{16}{3}$

$$9 = \frac{1}{9} + B; \quad 9 - \frac{1}{9} = \frac{80}{9} = B$$

$$= \frac{x^2}{2} - 3x + \frac{1}{9}\ln|x-1| + \frac{80}{9}\ln|x+2| + \frac{16}{3}\,\frac{1}{x+2} + C \quad //$$