Complex numbers - Exercises with detailed solutions

1. Compute real and imaginary part of $z = \dfrac{i-4}{2i-3}$.

2. Compute the absolute value and the conjugate of
$$z = (1+i)^6, \qquad w = i^{17}.$$

3. Write in the "algebraic" form $(a+ib)$ the following complex numbers
$$z = i^5 + i + 1, \qquad w = (3+3i)^8.$$

4. Write in the "trigonometric" form $(\rho(\cos\theta + i\sin\theta))$ the following complex numbers
$$a)8 \qquad b)6i \qquad c)\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)^7.$$

5. Simplify
   (a) $\dfrac{1+i}{1-i} - (1+2i)(2+2i) + \dfrac{3-i}{1+i}$;
   (b) $2i(i-1) + \left(\sqrt{3}+i\right)^3 + (1+i)\overline{(1+i)}$.

6. Compute the square roots of $z = -1-i$.

7. Compute the cube roots of $z = -8$.

8. Prove that there is no complex number such that $|z| - z = i$.

9. Find $z \in \mathbb{C}$ such that
$$a)\overline{z} = i(z-1) \qquad b)z^2 \cdot \overline{z} = z \qquad c)|z+3i| = 3|z|.$$

10. Find $z \in \mathbb{C}$ such that $z^2 \in \mathbb{R}$.

11. Find $z \in \mathbb{C}$ such that
    (a) $\operatorname{Re}(z(1+i)) + z\overline{z} = 0$;
    (b) $\operatorname{Re}(z^2) + i\operatorname{Im}(\overline{z}(1+2i)) = -3$;
    (c) $\operatorname{Im}((2-i)z) = 1$.

12. Find $a \in \mathbb{R}$ such that $z = -i$ is a root for the polynomial $P(z) = z^3 - z^2 + z + 1 + a$. Furthermore, for such value of a find the factors of $P(z)$ in $\mathbb{R}$ and in $\mathbb{C}$.

Solutions

1. $z = \dfrac{i-4}{2i-3} = \dfrac{i-4}{2i-3} \cdot \dfrac{2i+3}{2i+3} = \dfrac{-2+3i-8i-12}{-4-9} = \dfrac{14}{13} + i\dfrac{5}{13}$ hence $\mathrm{Re}\,(z) = \dfrac{14}{13}$ and $\mathrm{Im}\,(z) = \dfrac{5}{13}$.

2. $z = (1+i)^6 = \left(\sqrt{2}(\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4})\right)^6 = 8\left(\cos\dfrac{3\pi}{2} + i\sin\dfrac{3\pi}{2})\right) = -8i$. Hence $|z| = 8$ and $\bar{z} = 8i$.

   $w = i^{17} = i \cdot i^{16} = i \cdot \left(i^4\right)^4 = i \cdot (1)^4 = i$. Hence $|w| = 1$ and $\bar{w} = -i$.

3. $i^2 = -1$, $i^3 = -i$, $i^4 = 1$ e $i^5 = i$ then $z = i + i + 1 = 1 + 2i$.

   For w, we write $3+3i$ in the trigonometric form. We have $3+3i = 3\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$, hence

$$(3+3i)^8 = 3^8 \cdot 2^4 \left(\cos\left(8\cdot\frac{\pi}{4}\right) + i\sin\left(8\cdot\frac{\pi}{4}\right)\right) = 16\cdot 3^8\left(\cos 2\pi + i\sin 2\pi\right) = 16\cdot 3^8.$$

4. If $z = a+ib$, $a, b \in \mathbb{R}$, its trigonometric form is

$$z = \rho\left(\cos\theta + i\sin\theta\right), \qquad \text{where } \rho := \sqrt{a^2+b^2} \text{ and } \theta \text{ is such that } \cos\theta = \frac{a}{\rho},\ \sin\theta = \frac{b}{\rho}.$$

   a) $a = 8$, $b = 0$, $\cos\theta = 1$ e $\sin\theta = 0$. Hence $8 = 8\left(\cos 0 + i\sin 0\right)$.
   b) $6i = 6\left(0+i\right) = 6\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$.
   c) We use the de Moivre's Formula:

$$\left(\cos(\frac{\pi}{3}) - i\sin(\frac{\pi}{3})\right)^7 = \cos\frac{7\pi}{3} - i\sin\frac{7\pi}{3} = \cos 2\pi + \frac{\pi}{3} - i\sin 2\pi + \frac{\pi}{3} = \cos\frac{\pi}{3} - i\sin\frac{\pi}{3}.$$

5. (a) We compute

$$\frac{1+i}{1-i} - (1+2i)(2+2i) + \frac{3-i}{1+i} = \frac{1+i}{1-i}\cdot\frac{1+i}{1+i} - (1+2i)(2+2i) + \frac{3-i}{1+i}\cdot\frac{1-i}{1-i}$$
$$= i - 2 - 2i - 4i + 4 + \frac{3-1-3i-i}{2} = i + 2 - 6i + \frac{2-4i}{2} = 2 - 5i + 1 - 2i = 3 - 7i.$$

   (b) Since

$$\left(\overline{\sqrt{3}+i}\right)^3 = \left(\sqrt{3}-i\right)^3 = \left(\sqrt{3}-i\right)^2\left(\sqrt{3}-i\right) = \left(3-1-2i\sqrt{3}\right)\left(\sqrt{3}-i\right)$$
$$= \left(2-2i\sqrt{3}\right)\left(\sqrt{3}-i\right) = 2\sqrt{3} - 2i - 6i - 2\sqrt{3} = -8i,$$

   we obtain

$$2i(i-1) + \left(\overline{\sqrt{3}+i}\right)^3 + (1+i)\overline{(1+i)} = -2 - 2i - 8i + 2 = -10i.$$

6. Every $z \in \mathbb{C}$ has n distinct roots of order n, which correspond (in the complex plane) to the vertices of a regular n-agon inscribed in the circle of radius $\sqrt[n]{|z|}$ centered at the origin.

   When $z = \rho\left(\cos\theta + i\sin\theta\right) = \rho e^{i\theta}$, then the roots of order n of z are

$$\sqrt[n]{\rho}\left(\cos\left(\frac{\theta+2k\pi}{n}\right) + i\sin\left(\frac{\theta+2k\pi}{n}\right)\right) = \sqrt[n]{\rho}\,e^{\frac{\theta+2k\pi}{n}}. \quad k = 0, 1, 2, \ldots, n-1.$$

   The square roots of $z = -1-i = \sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$ are

$$z_1 = \sqrt[2]{\sqrt{2}}\left(\cos\left(\frac{\frac{5\pi}{4}}{2}\right) + i\sin\left(\frac{\frac{5\pi}{4}}{2}\right)\right) = \sqrt[4]{2}\left(\cos\frac{5\pi}{8} + i\sin\frac{5\pi}{8}\right) \quad \text{and}$$

$$z_2 = \sqrt[2]{\sqrt{2}}\left(\cos\left(\frac{\frac{5\pi}{4}+2\pi}{2}\right)+i\sin\left(\frac{\frac{5\pi}{4}+2\pi}{2}\right)\right) = \sqrt[4]{2}\left(\cos\frac{13\pi}{8}+i\sin\frac{13\pi}{8}\right).$$

We could also argue as follows: the equation

$$(x+i\,y)^2 = -1-i$$

is equivalent to the system

$$\begin{cases} x^2-y^2=-1, \\ 2xy=-1, \end{cases}$$

which admits solutions

$$z = \pm\left(\sqrt{\frac{\sqrt{2}-1}{2}} - \frac{i}{2}\sqrt{\frac{2}{\sqrt{2}-1}}\right)$$

which coincide with z_1 and z_2.

7. The trigonometric form of $z=-8$, is $z = 8(\cos\pi + i\sin\pi)$. Then

$$z_1 = \sqrt[3]{8}\left(\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)\right) = 2\left(\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)\right) = 1+i\sqrt{3},$$

$$z_2 = \sqrt[3]{8}(\cos\pi + i\sin\pi) = 2(\cos\pi+i\sin\pi) = -2, \text{ and}$$

$$z_3 = \sqrt[3]{8}\left(\cos\left(\frac{5\pi}{3}\right)+i\sin\left(\frac{5\pi}{3}\right)\right) = 2\left(\cos\left(\frac{5\pi}{3}\right)+i\sin\left(\frac{5\pi}{3}\right)\right) = 1-i\sqrt{3}.$$

8. Suppose that some $z\in\mathbb{C}$ satisfies the equation. Then $|z| = \mathrm{Re}\,(z)+i(\mathrm{Im}\,(z)+1)$. Hence, since $|z|\in\mathbb{R}$, necessarily $\mathrm{Im}\,(z)=-1$. The equation then is $\sqrt{(\mathrm{Re}\,(z))^2+1} = \mathrm{Re}\,(z)$, and, squaring, we obtain $1=0$.

9. We will use the notation $z=a+ib$, $a,b\in\mathbb{R}$.

a) The equation becomes $a-ib = i(a+ib-1)$, that is $a-ib=-b+i(a-1)$. Then $a=-b$ and $-b=a-1$, which has no solution; We conclude that the equation has no solution.

b) The equation becomes $z\cdot(z\bar{z}-1)=0$. Hence a first solution is $z=0$, while the others satisfy $z\bar{z}=|z|^2=1$. Then also all the points of the circle of radius 1 centered at the origin satisfies the equation.

c) We square both terms and we obtain

$$|z+3i|^2 = |a+i(b+3)|^2 = a^2+(b+3)^2, \qquad (3|z|)^2 = 9(a^2+b^2).$$

Hence we have to solve the equation

$$a^2+(b+3)^2 = 9(a^2+b^2) \quad\Leftrightarrow\quad 8(a^2+b^2)=6b+9 \quad\Leftrightarrow\quad a^2+b^2-\frac{3}{4}b=\frac{9}{8} \quad\Leftrightarrow\quad a^2+\left(b-\frac{3}{8}\right)^2 = \left(\frac{9}{8}\right)^2.$$

Then the solution are all the points of the circle of radius $9/8$ centered at $(0,3/8)$.

10. If $z=a+ib$, $a,b\in\mathbb{R}$ then $z^2\in\mathbb{R}$ if and only if $a^2-b^2+2iab\in\mathbb{R}$, that is if and only if $ab=0$. Hence $z^2\in\mathbb{R}$ if and only if $z\in\mathbb{R}$ $(b=0)$ or if z is a pure imaginary number $(a=0)$.

11. Let $z=a+ib$, $a,b\in\mathbb{R}$.

   (a) $\mathrm{Re}\,(z(1+i)) = \mathrm{Re}\,((a+ib)(1+i)) = \mathrm{Re}\,(a-b+i(a+b)) = a-b$. The equation is then equivalent to

$$a-b+a^2+b^2=0 \quad\Leftrightarrow\quad \left(a+\frac{1}{2}\right)^2+\left(b-\frac{1}{2}\right)^2=\frac{1}{2}$$

   whose solutions are the points of the circle with center in $(-1/2,1/2)$ and radius $\sqrt{2}/2$.

(b) Since $z^2 = a^2 - b^2 + 2iab$ and $\bar{z}(1+2i) = (a-ib)(1+2i) = a+2b+i(2a-b)$, the equation can be written as

$$a^2 - b^2 + i(2a-b) = -3,$$

and we deduce $2a = b$ and $a^2 - b^2 = -3$. The solution of this system are $z_1 = 1+2i$ and $z_2 = -1-2i$, the unique solutions of the starting equation.

(c) Since $(2-i)(a+bib) = 2a+b+i(2b-a)$, the equation can be written as

$$2b - a = 1.$$

whose solutions are the points of the line $x - 2y + 1 = 0$.

12. If $z = -i$, then $z^2 = -1$, $z^3 = i$, and $P(-i) = i+1-i+1+a = 2+a$. Then $-i$ is a root for P if and only if $a = -2$. Since $P(z) = z^3 - z^2 + z - 1$ contains $z-1$, we have $P(z) = (z-1)(z-i)(z+i)$.