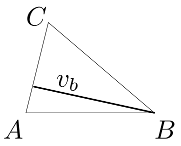
1. [6 bodov] Dané sú body $A = (0, 1, 2)$, $B = (1, -3, 3)$, $C = (-1, -1, 4)$. Vypočítajte

a) obsah trojuhulníka ABC, b) Vzdialenosť bodu B od priamky AC.



a) $\overrightarrow{AB} = B - A = (1, -4, 1)$, $\overrightarrow{AC} = C - A = (-1, -2, 2)$,

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -4 & 1 \\ -1 & -2 & 2 \end{vmatrix} = (-6, -3, -6), \quad \underline{P_\triangle} = \frac{1}{2}\|\overrightarrow{AB} \times \overrightarrow{AC}\| = \frac{1}{2}\sqrt{36 + 9 + 36} = \frac{1}{2}\sqrt{81} = \underline{\frac{9}{2}}.$$

b) $P_\triangle = \frac{1}{2}\|\overrightarrow{AC}\| v_b$, kde v_b je výška na stranu $b = AC$, t.j. vzdialenosť bodu B od priamky AC.

$$\frac{9}{2} = \frac{1}{2}\|\overrightarrow{AC}\| v_b \implies v_b = \frac{9}{\sqrt{1 + 4 + 4}} = \frac{9}{3} = \underline{3}\,.$$

2. [6 b.] $f(x) = \arcsin(2x + 1)$. Určte

a) $D(f)$, b) f' a $D(f')$, c) hodnotu $f(0)$, d) $\lim\limits_{x \to -1} f(x)$

a) $-1 \leq 2x + 1 \leq 1 \implies -2 \leq 2x \leq 0 \implies -1 \leq x \leq 0$, $\underline{D(f) = \langle -1, 0 \rangle}$

b) $\underline{\big(\arcsin(2x + 1)\big)' = \frac{1}{\sqrt{1 - (2x+1)^2}}(2x + 1)' = \frac{2}{\sqrt{1 - (2x+1)^2}}\,, \quad D(f') = (-1, 0).}$

c) $f(0) = \arcsin(1) = \underline{\frac{\pi}{2}}$. d) $\lim\limits_{x \to -1} f(x) = \arcsin(2 \cdot (-1) + 1) = \arcsin(-1) = \underline{-\frac{\pi}{2}}$.

3. [6 b.] Vypočítajte a) $\lim\limits_{x \to \infty} \dfrac{x^3 + 2x^2 + 2x}{2x^3 - x + 1}$, b) $\big(\operatorname{arctg} 2x\big)'$, c) $f'(1)$, ak $f(x) = \mathrm{e}^{(2x^2 - x - 1)}$, d) $\left(\dfrac{\ln x^2}{x}\right)'$

a) $$\lim_{x \to \infty} \frac{x^3 + 2x^2 + 2x}{2x^3 - x + 1} = \lim_{x \to \infty} \frac{x^3(1 + \frac{2}{x} + \frac{2}{x^2})}{x^3(2 - \frac{1}{x^2} + \frac{1}{x^3})} = \frac{1}{2}.$$

b) $$\big(\operatorname{arctg} 2x\big)' = \frac{1}{1 + (2x)^2}(2x)' = \frac{2}{1 + 4x^2},$$

c) $$\left(\mathrm{e}^{(2x^2 - x - 1)}\right)' = \mathrm{e}^{(2x^2 - x - 1)}(2x^2 - x - 1)' = (4x - 1)\mathrm{e}^{2x^2 - x - 1}, \; f'(1) = (4 \cdot 1 - 1)\mathrm{e}^{2 - 1 - 1} = 3\mathrm{e}^0 = 3.$$

d) $$\left(\frac{\ln x^2}{x}\right)' = \frac{\frac{1}{x^2}(2x)x - \ln x^2}{x^2} = \frac{2 - 2\ln x}{x^2}.$$

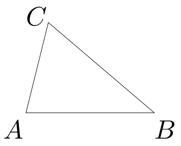
4. [2 b.] Určte $a \in R$, pre ktoré $\begin{vmatrix} 1 & a \\ a & 4 \end{vmatrix} = 0$. *[*prémia* 2 b.] $f(x) = \begin{cases} \frac{\sin x}{x} & \text{ak } x \neq 0\,, \\ 1 & \text{ak } x = 0 \end{cases}$, vypočítajte $f'(0)$.

$$\begin{vmatrix} 1 & a \\ a & 4 \end{vmatrix} = 4 - a^2 = 0 \implies \underline{a = \pm 2}.$$

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{\frac{\sin x}{x} - 1}{x} \frac{x}{x} = \lim_{x \to 0} \frac{\sin x - x}{x^2} \; \text{“}\tfrac{0}{0}\text{”}$$

$$= \lim_{x \to 0} \frac{\cos x - 1}{2x} \; \text{“}\tfrac{0}{0}\text{”} = \lim_{x \to 0} \frac{-\sin x}{2} = 0 \; \underline{f'(0) = 0}.$$

1. [6 bodov] Nájdite všeobecnú rovnicu roviny ρ, ktorá obsahuje body $A = (-1, 0, 1)$, $B = (0, -4, 2)$, $C = (-2, -2, 3)$ a zistite, či je trojuholník ABC pravouhlý.



a) $\overrightarrow{AB} = B - A = (1, -4, 1),\ \overrightarrow{AC} = C - A = (-1, -2, 2),$

$$\rho\colon \begin{vmatrix} x+1 & y & z-1 \\ 1 & -4 & 1 \\ -1 & -2 & 2 \end{vmatrix} = -6(x+1) - 3y - 6(z-1) = 0 \implies \underline{\rho\colon 2x + y + 2z = 0}$$

b) $\overrightarrow{AB} \cdot \overrightarrow{AC} = (1, -4, 1) \cdot (-1, -2, 2) = -1 + 8 + 2 = 9$ $\overrightarrow{BA} \cdot \overrightarrow{BC} = (-1, 4, -1) \cdot (-2, 2, 1) = 2 + 8 - 1 = 9$ $\overrightarrow{CA} \cdot \overrightarrow{CB} = (1, 2, -2) \cdot (2, -2, -1) = 2 - 4 + 2 = 0 \implies \triangle$ABC je pravopuhlý.

2. [6 b.] [6 b.] $f(x) = \arccos(2x + 1)$. Určte a) $D(f)$, b) f' a $D(f')$, c) hodnotu $f(-1)$, d) $\lim\limits_{x\to 0} f(x)$

a) $-1 \le 2x + 1 \le 1 \implies -2 \le 2x \le 0 \implies -1 \le x \le 0,$ $\underline{D(f) = \langle -1, 0\rangle}$

b) $f' = \frac{-1}{\sqrt{1-(2x+1)^2}}(2x+1)' = \frac{-2}{\sqrt{1-(2x+1)^2}}$, $D(f') = (-1, 0)$

c) $f(-1) = \arccos(-2 + 1) = \arccos(-1) = \pi$

d) $\lim\limits_{x\to 0} f(x) = f(0) = \arccos(1) = 0$

3. [6 b.] Vypočítajte a) $\lim\limits_{x\to\infty} \dfrac{x^2 + 2x + 1}{2x^3 - x + 1}$, b) $\big(\ln(x^2 + 1)\big)'$, c) $f'(1)$, ak $f(x) = \operatorname{arccotg} \mathrm{e}^x$, d) $\left(\dfrac{2^x}{x}\right)'$

a) $\lim\limits_{x\to\infty} \dfrac{x^2 + 2x + 1}{2x^3 - x + 1} = \lim\limits_{x\to\infty} \dfrac{x^2(1 + \frac{2}{x} + \frac{1}{x^2})}{x^3(2 - \frac{1}{x^2} + \frac{1}{x^3})} = 0$

b) $\big(\ln(x^2 + 1)\big)' = \frac{1}{x^2+1}(x^2 + 1)' = \frac{2x}{x^2+1}$

c) $(\operatorname{arccotg} \mathrm{e}^x)' = \dfrac{-1}{1 + (e^x)^2} e^x$, $f'(1) = \dfrac{-1}{1 + e^2} e = \dfrac{-e}{1 + e^2}$

d) $\left(\dfrac{2^x}{x}\right)' = \dfrac{2^x x \ln 2 - 2^x}{x^2}$

4. [2 b.] Pomocou $d = \begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix}$ vyjadrite $D = \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$. * [*prémia* 2 b.] $f(x) = \begin{cases} x \operatorname{arctg} \frac{1}{x}, & \text{ak } x \ne 0, \\ 0, & \text{ak } x = 0 \end{cases}$. Vypočítajte $f'(0)$

4. $D = -d$ (výmenou $R_2 \leftrightarrow R_3$ sa zmení znamienko determinantu.

*) $f'(0) = \lim\limits_{x\to 0} \dfrac{f(x) - f(0)}{x - 0} = \lim\limits_{x\to 0} \dfrac{x \operatorname{arctg} \frac{1}{x} - 0}{x} = \lim\limits_{x\to 0} \operatorname{arctg} \dfrac{1}{x}$ $\nexists$, lebo $\lim\limits_{x\to 0^+} \operatorname{arctg} \dfrac{1}{x} = \dfrac{\pi}{2}$, $\lim\limits_{x\to 0^-} \operatorname{arctg} \dfrac{1}{x} = -\dfrac{\pi}{2}$