Pr1A. [14 bodov] $f(x) = 2x^5 + x^3 + 2x^2 + x + 2,\ g(x) = x^4 + x^3 + x + 2 \in P(Z_3)$.
Určte $a(x), b(x) \in P(Z_3)$, pre ktoré $\quad \gcd(f, g) = a(x)f(x) + b(x)g(x)$

$$(2x^5 + x^3 + 2x^2 + x + 2) : (x^4 + x^3 + x + 2) = 2x + 1$$
$$-\underline{(2x^5 + 2x^4 + 2x^2 + x)}$$
$$x^4 + x^3 + 2$$
$$-\underline{(x^4 + x^3 + x + 2)}$$
$$2x = f_2 = f - (2x + 1)g$$
$$(x^4 + x^3 + x + 2) : (2x) = 2x^3 + 2x^2 + 2$$
$$-\underline{x^4}$$
$$x^3 + x + 2$$
$$-\underline{x^3}$$
$$x + 2$$
$$-\underline{x}$$
$$2 = \gcd(f, g) = g(x) - (2x^3 + 2x^2 + 2)f_2 = g(x) - (2x^3 + 2x^2 + 2)[f(x) - (2x + 1)g(x)]$$
$$\gcd(f, g) = \underbrace{(x^3 + x^2 + 1)}_{a(x)} f(x) + \underbrace{(x^4 + 2x^2 + x)}_{b(x)} g(x)$$

Pr1B. [14] $f(x) = 2x^5 + x^4 + 2x^2 + x + 2,\ g(x) = x^4 + x^3 + x + 2 \in P(Z_3)$.
Určte $a(x), b(x) \in P(Z_3)$, pre ktoré $\quad \gcd(f, g) = a(x)f(x) + b(x)g(x)$

$$(2x^5 + x^4 + 2x^2 + x + 2) : (x^4 + x^3 + x + 2) = 2x + 2$$
$$-\underline{(2x^5 + 2x^4 + 2x^2 + x)}$$
$$2x^4 + 2$$
$$-\underline{(2x^4 + 2x^3 + 2x + 1)}$$
$$x^3 + x + 1 = f_2 = f - (2x + 2)g = f + (x + 1)g$$
$$(x^4 + x^3 + x + 2) : (x^3 + x + 1) = x + 1$$
$$-\underline{(x^4 + x^2 + x)}$$
$$x^3 + 2x^2 + 2$$
$$-\underline{(x^3 + x + 1)}$$
$$2x^2 + 2x + 1 = f_3 = g - (x + 1)f_2$$
$$(x^3 + x + 1) : (2x^2 + 2x + 1) = 2x + 1$$
$$-\underline{(x^3 + x^2 + 2x)}$$
$$2x^2 + 2x + 1, \text{zv. } 0 \implies \gcd(f, g) = f_3$$
$$= g - (x + 1)f_2 = g - (x + 1)[f + (x + 1)g] = (2x + 2)f + [1 - (x + 1)^2]g$$
$$= \underbrace{(2x + 2)}_{a(x)} f(x) + \underbrace{(2x^2 + x)}_{b(x)} g(x)$$

Pr.2A [12 bodov] $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{pmatrix}$. Vypočítajte

a) trace$A$, b) vlastné čísla matice A, c) maticu P a diagonálnu maticu D, pre ktorú $A = PDP^{\top}$.

a) trace$A = 5$, b) Očividne $\lambda_1 = 1$ aj $\lambda_2 = 0$ sú vlastné čísla matice A, teda $\lambda_3 = 4$, prípadne počítame $\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 2-\lambda & 2 \\ 0 & 2 & 2-\lambda \end{vmatrix} = -\lambda(1-\lambda)(4-\lambda)$ a dostaneme $\sigma(A) = (1, 0, 4)$.

c) vlastné vektory: pre $\lambda_1 = 1$ je $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, pre $\lambda_2 = 0$ $\quad A - \lambda_2 I = A \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$,

$\lambda_3 = 4 \qquad A - \lambda_3 I = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 2 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $\|\mathbf{v}_2\| = \|\mathbf{v}_3\| = \sqrt{2}$, $\|\mathbf{v}_1\| = 1$.

$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}, \qquad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

Pr.2B [12 bodov] $A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{pmatrix}$. Vypočítajte

a) trace$A$, b) vlastné čísla matice A, c) maticu P a diagonálnu maticu D, pre ktorú $A = PDP^{\top}$.

a) trace$A = 3$, b) Očividne $\lambda_1 = -1$ aj $\lambda_2 = 0$ sú vlastné čísla matice A, teda $\lambda_3 = 4$, prípadne počítame $\begin{vmatrix} -1-\lambda & 0 & 0 \\ 0 & 2-\lambda & 2 \\ 0 & 2 & 2-\lambda \end{vmatrix} = \lambda(-1-\lambda)(\lambda-4)$ a dostaneme $\sigma(A) = (-1, 0, 4)$.

c) vlastné vektory: pre $\lambda_1 = -1$ je $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, pre $\lambda_2 = 0$ $\quad A - \lambda_2 I = A \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$,

$\lambda_3 = 4 \qquad A - \lambda_3 I = \begin{pmatrix} -5 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & 2 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $\|\mathbf{v}_2\| = \|\mathbf{v}_3\| = \sqrt{2}$, $\|\mathbf{v}_1\| = 1$.

$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}, \qquad D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix}$