$\forall m, n \in \mathbb{Z}: 2 \mid m \wedge 2 \mid n \Rightarrow 2 \mid (m+n)$

$2 \mid m \wedge 2 \mid n \Rightarrow m = 2k \wedge n = 2l, \; k, l \in \mathbb{Z} \Rightarrow$

$\Rightarrow m + n = 2k + 2l = 2(k+l) \Rightarrow 2 \mid (m+n)$

---

$\forall x, y \in \mathbb{R}: x \in \mathbb{Q} \wedge y \in \mathbb{Q} \Rightarrow x + y \in \mathbb{Q}$

$x \in \mathbb{Q} \wedge y \in \mathbb{Q} \Rightarrow x = \frac{p}{q}, y = \frac{r}{s}, \; p, q, r, s \in \mathbb{Z} \Rightarrow$

$\Rightarrow x + y = \frac{p}{q} + \frac{r}{s} = \frac{ps + qr}{qs} = \frac{m}{n}, \; m, n \in \mathbb{Z} \Rightarrow$

$\Rightarrow x + y \in \mathbb{Q}$

---

$X, Y, Z$ sú množiny

$X \subseteq Y \Rightarrow X \cup Z \subseteq Y \cup Z$

$X \cup Z \subseteq Y \cup Z$

$x \in X \cup Z \Rightarrow x \in Y \cup Z$

$\underline{x \in X \cup Z} \Rightarrow x \in X \vee x \in Z$

1) $x \in X \Rightarrow x \in Y \Rightarrow x \in Y \cup Z$

2) $x \in Z \Rightarrow x \in Y \cup Z$

$\Rightarrow \underline{x \in Y \cup Z}$

---

$\forall n \in \mathbb{Z}: 5 \nmid n^3 \Rightarrow 5 \nmid n$

$\exists n \in \mathbb{Z}: 5 \mid n^3 \wedge 5 \nmid n$

$\underline{5 \nmid n^3 \wedge 5 \mid n}$

$\forall n \in \mathbb{Z}: 5 \mid n \Rightarrow 5 \mid n^3$

$5 / n \Rightarrow n = 5k,\ k \in \mathbb{Z} \Rightarrow n^3 = 125k^3 =$

$= 5 \cdot 25k^3 \Rightarrow 5 / n^3$

---

$\forall x, y, z \in \mathbb{Z} : (x+y+z) \geq 3 \Rightarrow (x \geq 1 \vee y \geq 1 \vee z \geq 1)$

$p \Rightarrow q \sim \neg q \Rightarrow \neg p \qquad \bar{q} \Rightarrow \bar{p}$

$x < 1 \wedge y < 1 \wedge z < 1 \overset{?}{\Rightarrow} x+y+z < 3$

$x < 1 \wedge y < 1 \wedge z < 1 \Rightarrow x+y+z < 1+1+1 \Rightarrow$

$\Rightarrow x+y+z < 3$

---

$X \qquad X \times \emptyset = \emptyset$

sporom

Nech $X \times \emptyset \neq \emptyset \Rightarrow \exists m \in X \times \emptyset \Rightarrow$

$\Rightarrow m = (x, y) \in X \times \emptyset \Rightarrow x \in X \wedge \underline{y \in \emptyset} \Rightarrow$ spor

---

100 lôpt 9 krabíc aspoň v 1 krabici

bude aspoň 12 lôpt

sporom

Nech v každej krabici je najviac 11 lôpt.

Potom spolu je najviac $9 \cdot 11 = 99$ lôpt, $99 < 100$ spor

---

$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + n(n+1) = \dfrac{n(n+1)(n+2)}{3} \quad V(n)$

$$\sum_{i=1}^{n} i(i+1) = \frac{n(n+1)(n+2)}{3}$$

$1^\circ$ $n=1$ $\qquad 1\cdot 2 = \frac{1\cdot 2\cdot 3}{3}$

$2^\circ$ $\forall n \in N : V(n) \Rightarrow V(n+1)$

$$\underbrace{1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \dots + n(n+1)} + (n+1)(n+2) =$$

$$= \frac{n(n+1)(n+2)}{3} + (n+1)(n+2) =$$

$$= (n+1)(n+2)\left[\frac{n}{3} + 1\right] = (n+1)(n+2)\frac{n+3}{3} =$$

$$= \frac{(n+1)(n+2)(n+3)}{3}$$

$3^\circ$ : $1^\circ \wedge 2^\circ \Longrightarrow \forall n \in N : V(n)$

---

$$1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{1}{6} n(n+1)(2n+1) \qquad V(n) \quad n \geq 1$$

$1^\circ$ $n=1$ $\qquad 1^2 = \frac{1}{6} 1\cdot(1+1)(2\cdot 1+1)$

$2^\circ$ $\forall n \geq 1 : V(n) \Rightarrow V(n+1)$

$$\underbrace{1^2 + 2^2 + 3^2 + \dots + n^2} + (n+1)^2 \stackrel{?}{=} \frac{1}{6}(n+1)(n+2)(2n+3)$$

$$\frac{1}{6} n(n+1)(2n+1) + (n+1)^2 \stackrel{?}{=} \frac{1}{6}(n+1)(n+2)(2n+3)$$

$$L^v = \frac{1}{6} n\left(2n^2 + n + 2n + 1\right) + n^2 + 2n + 1 =$$

$$= \frac{1}{6}(2n^3 + 3n^2 + n) + \frac{1}{6}(6n^2 + 12n + 6) =$$

$$= \frac{1}{6}(2n^3 + 9n^2 + 13n + 6)$$

$$P = \frac{1}{6}(n+1)(n+2)(2n+3) = \frac{1}{6}(n+1)(2n^2+3n+4n+6) =$$

$$= \frac{1}{6}(n+1)(2n^2+7n+6) = \frac{1}{6}(2n^3+7n^2+6n+2n^2+7n+6) =$$

$$= \frac{1}{6}(2n^3+9n^2+13n+6)$$

$$L = P$$

$3^\circ$ $1^\circ \wedge 2^\circ \Rightarrow \forall n \geq 1 : V(n)$

---

$$4 \mid (6\cdot 7^n - 2\cdot 3^n) \qquad n \geq 1$$

$1^\circ$ $n=1$ $\quad 6\cdot 7^1 - 2\cdot 3^1 = 42 - 6 = 36 \qquad 4 \mid 36$

$2^\circ$ $\forall n \geq 1 : V(n) \Rightarrow V(n+1)$

$$6\cdot 7^{n+1} - 2\cdot 3^{n+1} = 7\cdot 6\cdot 7^n - 3\cdot 2\cdot 3^n =$$

$$= (8-1)6\cdot 7^n - (4-1)2\cdot 3^n = 8\cdot 6\cdot 7^n - 6\cdot 7^n - 4\cdot 2\cdot 3^n +$$

$$+ 2\cdot 3^n = \underline{8\cdot 6\cdot 7^n} - \underline{4\cdot 2\cdot 3^n} - (\underline{6\cdot 7^n - 2\cdot 3^n})$$

$3^\circ$ $1^\circ \wedge 2^\circ \Rightarrow \forall n \geq 1 : V(n)$

---

$$2n+1 \leq 2^n \qquad n \geq 3$$

$1^\circ$ $2\cdot 3+1 \leq 2^3 \qquad 7 \leq 8$

$2^\circ$ $2^{n+1} = 2\cdot 2^n = 2^n + 2^n \geq 2^n + 2 \geq 2n+1+2 =$

$$= 2n+3 = 2(n+1)+1$$

$3^\circ$ $1^\circ \wedge 2^\circ \Rightarrow \forall n \geq 3 : V(n)$