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# A REVERSED VON NEUMANN THEOREM

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Abstract. A densely defined operator $T$ acting between Hilbert spaces is shown to be closed if and only if $T^*T$ and $TT^*$ are both selfadjoint operators on the corresponding Hilbert spaces. This is an extension of the classical von Neumann theorem [2] on the selfadjointness of $T^*T$ whenever $T$ is closed.

## 1. Closedness of Hilbert space operators

**Theorem 1.** *Let $\mathfrak{H}$, $\mathfrak{K}$ be real or complex Hilbert spaces and $T : \mathfrak{H} \to \mathfrak{K}$ be densely defined linear operator. Assume that the operators $T^*T$ and $TT^*$ on the Hilbert spaces $\mathfrak{H}$ and $\mathfrak{K}$, respectively, are selfadjoint operators. Then $T$ is necessarily closed.*

*Proof.* By assumption, the operators $I + T^*T$ and $I + TT^*$ are selfadjoint, surjective operators in the corresponding Hilbert spaces, that is

$$\operatorname{ran}(I + T^*T) = \mathfrak{H} \qquad \text{and} \qquad \operatorname{ran}(I + TT^*) = \mathfrak{K}. \tag{1}$$

Here, the symbol $I$ stands for the identity operator of the corresponding Hilbert space. As a simple consequence, the operator $TT^*$ is densely defined, therefore $T^*$ is also densely defined. Hence $T$ is closable and its closure is exactly $T^{**}$. Taking into account that $T^{**}$ extends $T$ at the same time, the closedness of $T$ is equivalent to the statement $\operatorname{dom} T^{**} = \operatorname{dom} T$. Therefore we check that for if $z$ belongs to $\operatorname{dom} T^{**}$ then $z$ is from $\operatorname{dom} T$ in any case. In view of the identities in (1), we find (unique) $u$ and $v$ from $\operatorname{dom} T^*T$ and $\operatorname{dom} TT^*$, respectively, such that the following identities hold:

$$\begin{cases} z = u + T^*Tu, \\ T^{**}z = v + TT^*v. \end{cases} \tag{2}$$

Of course, $u$ and $v$ belong to $\operatorname{dom} T$ and $\operatorname{dom} T^*$ as well, respectively, so we have at once that

$$T^{**}z = T^{**}u + T^{**}T^*Tu = Tu + T^{**}T^*Tu.$$

At the same time, $T^{**}T^*$, as symmetric operator, extends the selfadjoint operator $TT^*$, hence they are equal: $TT^* = T^{**}T^*$. Therefore

$$T^{**}z = Tu + TT^*Tu = (I + TT^*)Tu$$

holds also true. Taking into account of the second identity in (2), we have the following identity:

$$0 = (Tu - v) + TT^*(Tu - v).$$

Here $u$ belongs to $\operatorname{dom} T^*T$, therefore $Tu$ belongs to $\operatorname{dom} T^*$ as well as $v$, therefore we have that

$$0 = (Tu - v \,|\, Tu - v) + (TT^*(Tu - v) \,|\, Tu - v) = \|Tu - v\|^2 + \|T^*(Tu - v)\|^2.$$

The latter identity means, of course, that $Tu = v$, therefore, in view of identity (2) that

$$z - u = T^*Tu = T^*v \in \operatorname{dom} T.$$

Since $z = (z - u) + u$, where both components are from the domain of $T$, we conclude that $z$ belongs to $\operatorname{dom} T$ as well. The proof is therefore complete. □

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## 2. Revised von Neumann theorem

A direct consequence of Theorem 1 enables to formulate the classical von Neumann theorem in a form, expressing equivalence of closedness of the operator $T$ and the selfadjointness of the operators $T^*T$ and $TT^*$ on the corresponding Hilbert spaces. The last step in the proof of this statement is due to the following Lemma stating that a surjective symmetric operator is automatically selfadjoint, see also [6] for densely defined symmetric operators.

**Lemma 2.** *Let $S$ be a not necessarily densely defined symmetric operator on a Hilbert space $\mathfrak{H}$ with full range, i.e.* $\operatorname{ran} S = \mathfrak{H}$*. Then $S$ is automatically selfadjoint (and therefore clearly densely defined as well).*

*Proof.* First of all we show that the domain $\operatorname{dom} S$ is dense: for if $z$ is from $\{\operatorname{dom} S\}^{\perp}$ then $z = Sx$ for some $x$ from $\operatorname{dom} S$, thanks to the surjectivity of $S$. Therefore we have for each $u$ from $\operatorname{dom} S$ that

$$0 = (z \,|\, u) = (Sx \,|\, u) = (x \,|\, Su),$$

consequently, $x$ belongs to $\{\operatorname{ran} S\}^{\perp} = \{0\}$, i.e. $x = 0$. Hence we have, of course, $z = Sx = 0$, indeed. So $S^*$ exists and extends the symmetric operator $S$. In order to prove selfadjointness on $S$ it suffices therefore to show that $\operatorname{dom} S = \operatorname{dom} S^*$. Let therefore be $z$ from $\operatorname{dom} S^*$ and find some $x$ from $\operatorname{dom} S$ such that $S^*z = Sx = S^*x$. Then we have at the same time

$$z - x \in \ker S^* = \{\operatorname{ran} S\}^{\perp} = \{0\},$$

so that $z = x \in \operatorname{dom} S$, as claimed. □

**Theorem 3.** *Let $\mathfrak{H}$ and $\mathfrak{K}$ be real or complex Hilbert spaces and let $T : \mathfrak{H} \to \mathfrak{K}$ be densely defined operator. The following assertions are equivalent:*

(i) *$T$ is closed operator,*
(ii) *$T^*T$ and $TT^*$ are (positive) selfadjoint operators.*

*Proof.* The implication (ii)⇒(i) appears in Theorem 1. That (i) implies (ii) is known as von Neumann theorem. For the sake of completeness we include a simple proof. Assuming that $T$ is closed, in other words that its graph is a closed subspace in the product Hilbert space $\mathfrak{H} \times \mathfrak{K}$, we have at once the orthogonal decomposition into closed subspaces as follows:

$$\{(x, Tx) \,|\, x \in \operatorname{dom} T\} \oplus \{(-T^*z, z) \,|\, z \in \operatorname{dom} T^*\} = \mathfrak{H} \times \mathfrak{K}.$$

For each $u$ and $v$ from $\mathfrak{H}$ and $\mathfrak{K}$, respectively, we have (unique) $x$ and $z$ from $\operatorname{dom} T$ and $\operatorname{dom} T^*$, respectively, that the following identities hold:

$$\begin{cases} u = x - T^*z, \\ v = Tx + z. \end{cases}$$

The choices $v = 0$ and $u = 0$ imply that

$$-z = Tx \qquad \text{and} \qquad x = T^*z$$

holds true, respectively. This means that $Tx$ and $T^*z$ belong to $\operatorname{dom} T^*$ and $\operatorname{dom} T$, respectively, and as well, respectively, that

$$u = x + T^*Tx \qquad \text{and} \qquad v = TT^*z + z.$$

We have therefore proved that both $I + T^*T$ and $I + TT^*$ are surjective symmetric operators, and consequently selfadjoint ones, in virtue of Lemma 2. □

## 3. Normality of Hilbert space operators

One more application of Theorem 1 gives a closedness free form of normality of an unbounded operator on a Hilbert space as follows, see [4, Proposition 5.1.10].

**Theorem 4.** *For a densely defined operator $T$ in a (real or complex) Hilbert space $\mathfrak{H}$ the following conditions are equivalent:*

(i) $\operatorname{dom} T = \operatorname{dom} T^*$ *and* $\|Tx\| = \|T^*x\|$ *for every* $x \in \operatorname{dom} T$,
(ii) $T^*T = TT^*$ *are selfadjoint operators on* $\mathfrak{H}$.

*Proof.* In order to prove that (i) implies (ii) it suffices to prove that $T$ is closed. To see this, consider the following correspondence

$$(x, Tx) \mapsto (x, T^*x), \qquad x \in \operatorname{dom} T,$$

which defines a unitary operator between the graphs $G(T)$ and $G(T^*)$ of $T$ and $T^*$, respectively. Since an adjoint operator is always closed, $G(T^*)$ is complete and therefore that $G(T)$ is complete too, hence $T$ is closed. As a consequence of von Neumann's theorem (Theorem 3), $T^*T$ and $TT^*$ are selfadjoint operators.

By assuming (ii), the closedness of $T$ follows from Theorem 1. For the remainder of the proof of this implication we refer to [4, Proposition 5.1.10]. □

**Remark 5.** An operator $T$ satisfying the conditions of Theorem 4 is called, as well known, normal. Our revision consists of omitting the closedness of $T$ in the commonly known form of the definition of normality of unbounded Hilbert space operators.

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