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THE MODEL FOR COMMUTING, COMPLETELY NON DOUBLY COMMUTING, COMPATIBLE PAIRS OF ISOMETRIES.

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1. Preliminaries

Let H be a complex Hilbert space. Denote by $L(H)$ an algebra of all linear bounded operators on H. Recall the classical Wold's result [7].

Theorem 1. *Let $V \in L(H)$ be an isometry. There is a unique decomposition of H into orthogonal, reducing for V subspaces*

$$H = H_u \oplus H_s,$$

such that $V|_{H_u}$ is a unitary operator, $V|_{H_s}$ is a unilateral shift. Moreover

$$H_u = \bigcap_{n\geq 0} V^n H, \quad H_s = \bigoplus_{n\geq 0} V^n \ker V^*.$$

□

Suppose $V_1, V_2 \in L(H)$ are commuting isometries. We say that they *doubly commute*, if also $V_1^* V_2 = V_2 V_1^*$. The third author showed that for a doubly commuting pairs of isometries there is a natural extension of the Wold result to a pair [5] and showed a model of a pair of doubly commuting unilateral shift.. Similar decomposition has been obtained by the same authors when wandering subspaces are finite dimensional [1]. In the talk we will give a model for much bigger class of pairs of commuting, compatible isometries.

Denote $P_i^m := V_i^m V_i^{*m}$ for $i = 1, 2$. Recall that P_i^m is an orthogonal projection on $\mathcal{R}(V_i^n)$. A pair of commuting isometries $V_1, V_2 \in L(H)$ is called *compatible* if $P_1^n P_2^m = P_2^m P_1^n$ for every $n, m \in \mathbb{Z}_+$ (see [2], [3] and [6]). Since doubly commuting pairs of isometries are described in mentioned result, we focus on compatible, completely non doubly commuting pairs. We define two types of compatible pairs of isometries which gives the model.

2. Diagrams

One type compatible pairs of isometries are pairs given by a diagram. Recall the following:

Definition 1. The $J \subset \mathbb{Z}^2$ is called a diagram (in $\mathbb{Z}^2$) if for any $g \in (\mathbb{Z}_+ \cup \{0\})^2$ and any $j \in J$ the element $g + j$ belongs to J [3].

Every diagram generates a pair of isometries in the following way:

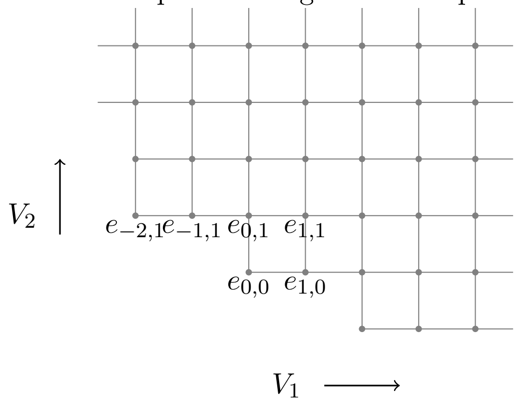
Example 1. Let us fix a diagram J in $\mathbb{Z}^2$ and orthonormal vectors $\{e_{i,j}\}_{(i,j)\in J}$ in a complex Hilbert space. We can define a new Hilbert space

$$H = \bigoplus_{(i,j)\in J} \mathbb{C} e_{i,j}$$

and isometries

$$V_1(e_{i,j}) = e_{i+1,j}, \quad V_2(e_{i,j}) = e_{i,j+1}.$$

See an example of a diagram in the picture.



Note that for a chosen $e_{i,j}$ and any nonnegative pairs of integers $(m,n) \neq (k,l)$ holds $V_1^n V_2^m e_{i,j} = e_{i+m,j+n}$ is orthogonal to $V_1^k V_2^l e_{i,j} = e_{i+k,j+l}$.

Definition 2. [4] A pair of isometries given by a diagram $J = \mathbb{Z}^2 \setminus \mathbb{Z}^2_-$ is called modified bi-shift.

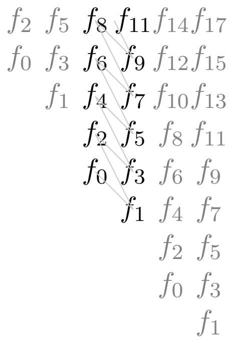
Remark 2. [4] *A completely non doubly commuting pair of isometries such that* $\ker V_1^* \cap \ker V_2^* = \{0\}$ *is a modified bi-shift.*

3. Generalized powers

We are going to define a class of pairs of isometries fulfilling condition $V_1^n = UV_2^m$ for some unitary operator U. Thus they will be called "generalized powers". For convenance we start with powers $n = 3, m = 2$.

Example 2. Let be given an unilateral shift V of multiplicity 1 on a Hilbert space $H = \bigoplus_{i \geq 0} \mathbb{C} f_i$ where $V f_i = f_{i+1}$. Define a pair of isometries $V_1 = V^2, V_2 = V^3$.

Since $V_1^3 = V_2^2$ it can not be chosen vectors $e_{i,j}$ to obtain a diagram. However there are some similarities with a diagram. Let us show it in the picture.



We can see in the picture that V_1 shifts vectors "horizontally" while V_2 shifts vectors "vertically". Consider the following notation of vectors:

$$\begin{cases} e_{2k,-i/2-3k} = f_i & \text{for } i \text{ odd, } k \in \mathbb{Z} \\ e_{1+2k,(i+1)/2-3k} = f_i, & \text{for } i \text{ even, } k \in \mathbb{Z}. \end{cases}$$

Isometries V_1, V_2 acts similar to pair given by a diagram on vectors $e_{i,j}$. The important difference is that not all vectors $e_{i,j}$ are orthogonal.

Let be given a unitary operator $U \in L(\mathcal{H})$ and vector $f \in \mathcal{H}$ such that the minimal U reducing subspace containing f is the whole $\mathcal{H}$. Then define a new Hilbert space $H = \bigoplus_{n \geq 0} H_n$ where $H_n = \mathcal{H}$. Denote $f_n = (0, \dots 0, f, 0, \dots) \in H$ where non zero value is on n-th coordinate. Define vectors $e_{i,j}$ in the following way:

$$\begin{cases} e_{2k,i/2-3k} = U^k f_i & \text{for } i \text{ odd, } k \in \mathbb{Z} \\ e_{1+2k,-(i+1)/2-3k} = U^k f_i, & \text{for } i \text{ even, } k \in \mathbb{Z}. \end{cases}$$

For $V_1 e_{ij} = e_{i+1,j}, V_2 e_i j = e_{i,j+1}$ we obtain $V_1^3 = UV_2^2$.

Note that condition $V_1^3 = UV_2^2$ do not guarantee that isometries are suitable powers of the same unilateral shift. Modify Example 2 by removing vector f_1. It can be done since $H \ominus \mathbb{C} f_1$ is invariant for V_1, V_2. A subspace $\ker V_1^* \cap \ker V_2^* = \{f_0\}$. Such a subspace would be 2 dimensional if V_1, V_2 were suitable powers of the same unilateral shift.

The border of a diagram is the set $\{(i,j) \in J : e_{ij} \in \ker V_1^* V_2^*\} \subset \mathbb{Z}^2$. The border can be defined as a sequence of pairs of positive integers giving "length" and "hight" of suitable parts of the border. The border can be defined also for the generalized powers as a finite sequence. In the following definition of generalized powers a (finite) sequence of pairs of integers is the border.

Definition 3. Let be given:

(1) a finite sequence of integers $\{(m_\alpha, n_\alpha)\}_{\alpha=0}^k$.
(2) unitary operator U on a Hilbert space $\mathcal{H}$ such that the minimal U reducing subspace containing some vector e is the whole $\mathcal{H}$.

Then define:

(1) $m := \sum m_\alpha, n := \sum n_\alpha$,
(2) set $I := \{(i,j) : m_1 + \dots m_\beta \leq i < m_1 + \dots m_{\beta+1}$ and $j \leq n_1 + \dots n_\beta\}$,
(3) Hilbert space $H := \bigoplus_{(i,j)\in I} H_{i,j}$ where $H_{i,j} = \mathcal{H}$,
(4) $e_{i,j} \in H$ a vector having e in (i,j) coordinate and 0 on remaining for $(i,j) \in I$,
(5) $e_{i+lm,j-ln} := U^l e_{i,j}$ for $(i,j) \in I$ and $l \in \mathbb{Z}$,
(6) J is a set for which has been defined vectors $e_{\cdot,\cdot}$.

Then set J is a diagram and we can define isometries $V_1(e_{i,j}) = e_{i+1,j},\ V_2(e_{i,j}) = e_{i,j+1}$. Such defined isometries fulfills equality $V_1^n = UV_2^m$ and are called *generalized powers*.

Note that in the definition of generalized powers it is not enough to give powers m, n but we need to give the full sequences.

4. Decomposition theorem

We obtain a model of compatible isometries by the following theorem:

Theorem 3. *Any non doubly commuting compatible part of commuting isometries is a orthogonal sum of isometries defined by diagrams and generalized powers.*

Note that generalized powers and pairs of isometries given by diagrams are not distinct sets. Let in the definition of generalized powers vector e is wandering for a unitary operator U. It means that U is a bilateral shift and $\mathcal{H} = \bigoplus_{n\in\mathbb{Z}} U^n(\mathbb{C}e)$. Then it is a pair of isometries given by a diagram which border is periodical sequence with period equal to the sequence from the definition of isometries as generalized powers.

References

[1] Z.Burdak, M. Kosiek, M.Słociński *The canonical Wold decomposition of commuting isometries with finite dimensional wandering spaces.* Bulletin des Sciences Mathématiques **137**, 653 – 658 (2013)

[2] Karel Horák; Vladimir Müller *On the structure of commuting isometries* Commentationes Mathematicae Universitatis Carolinae, **Vol. 28 (1987), No. 1, 165-171.**

[3] Karel Horák; Vladimir Müller *Functional model for commuting isometries* Czechoslovak Mathematical Journal, **Vol. 39 (1989), No. 2, 370-379.**

[4] Dan Popovici *A Wold-type decomposition for commuting isometric pairs* Proceedings of the American Mathematical Society **Vol. 132, No. 8, 2303-2314**

[5] Marek Słociński *On the Wold-type decomposition of a pair of commuting isometries* Annales Polonici Mathematici **XXXVII, (1980), 255-262.**

[6] Marek Słociński *Models of commuting contractions and isometries*, Report of the 11th Conference on Operator Theory Bucharest. 1986

[7] H. Wold, *A study in the analysis of stationary time series*, Uppsala 1954.