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STABILIZATION OF THE GEAR–GRIMSHAW SYSTEM ON A PERIODIC DOMAIN

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1. Introduction

The goal of this paper is to investigate the decay properties of the initial-value problem

$$\begin{cases} u' + uu_x + u_{xxx} + a_3 v_{xxx} + a_1 v v_x + a_2 (uv)_x + k(u - [u]) = 0, \\ b_1 v' + r v_x + v v_x + v_{xxx} + b_2 a_3 u_{xxx} + b_2 a_2 u u_x \\ \qquad\qquad\qquad\qquad + b_2 a_1 (uv)_x + k(v - [v]) = 0, \\ u(0,x) = \phi(x), \\ v(0,x) = \psi(x) \end{cases} \tag{1.1}$$

with periodic boundary conditions. In (1.1), $r, a_1, a_2, a_3, b_1, b_2, k$ are given real constants with $b_1, b_2, k > 0$, $u(t,x), v(t,x)$ are real-valued functions of the time and space variables $t \geq 0$ and $0 \leq x \leq 1$, the subscript x and the prime indicate the partial differentiation with respect to x and t, respectively, and $[f]$ denotes the mean value of f defined by

$$[f] := \int_0^1 f(x)\, dx.$$

When $k = 0$, system was proposed by Gear and Grimshaw [8] as a model to describe strong interactions of two long internal gravity waves in a stratified fluid, where the two waves are assumed to correspond to different modes of the linearized equations of motion. It has the structure of a pair of KdV equations with both linear and nonlinear coupling terms and has been object of intensive research in recent years. In what concerns the stabilization problems, most of the works have been focused on a bounded interval with a localized internal damping (see, for instance, [14] and the references therein). In particular, we also refer to [1] for an extensive discussion on the physical relevance of the system and to [3, 4, 5, 6, 7] for the results used in this paper.

We can (formally) check that the total energy

$$E = \frac{1}{2} \int_0^1 b_2 u^2 + b_1 v^2 \; dx$$

associated with the model satisfies the inequality

$$E' = -k \int_0^1 b_2 (u - [u])^2 + (v - [v])^2 \; dx \leq 0$$

in $(0, \infty)$, so that the energy in nonincreasing. Therefore, the following basic questions arise: are the solutions asymptotically stable for t sufficiently large? And if yes, is it possible to find a rate of decay? The aim of this paper is to answer these questions.

More precisely, we prove that for any fixed integer $s \geq 3$, the solutions are exponentially stable in the Sobolev spaces

$$H_p^s(0,1) := \{u \in H^s(0,1) \; : \; \partial_x^n u(0) = \partial_x^n u(1), \quad n = 0, \dots, s\}$$

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with periodic boundary conditions. This extends an earlier theorem of Dávila in [6] for $s \le 2$.

Before stating the stabilization result mentioned above, we first need to ensure the well posedness of the system. This was addressed by Dávila in [3] (see also [4]) under the following conditions on the coefficients:

$$\begin{gathered} a_3^2 b_2 < 1 \text{ and } r = 0 \\ b_2 a_1 a_3 - b_1 a_3 + b_1 a_2 - a_2 = 0 \\ b_1 a_1 - a_1 - b_1 a_2 a_3 + a_3 = 0 \\ b_1 a_2^2 + b_2 a_1^2 - b_1 a_1 - a_2 = 0. \end{gathered} \tag{1.2}$$

Indeed, under conditions (1.2), Dávila and Chaves [7] derived some conservation laws for the solutions of (1.1). Combined with an approach introduced in [2, 17], these conservation laws allow them to establish the global well-posedness in $H_p^s(0,1)$, for any $s \ge 0$. Moreover, the authors also give a simpler derivation of the conservation laws discovered by Gear and Grimshaw, and Bona et al [1]. We also observe that these conservation properties were obtained employing the techniques developed in [13] for the single KdV equation; see also [12].

The well-posedness result reads as follows:

Theorem 1.1. *Assume that condition* (1.2) *holds. If* $\phi, \psi \in H_p^s(0,1)$ *for some integer* $s \ge 3$*, then the system* (1.1) *has a unique solution satisfying*

$$u, v \in C([0,\infty); H_p^s(0,1)) \cap C^1([0,\infty); H_p^{s-3}(0,1)).$$

Moreover, the map $(\phi,\psi) \mapsto (u,v)$ *is continuous from* $\left(H_p^s(0,1)\right)^2$ *into*

$$\left(C([0,\infty); H_p^s(0,1)) \cap C^1([0,\infty); H_p^{s-3}(0,1))\right)^2.$$

For $k = 0$, the analogous theorem on the whole real line $-\infty < x < \infty$ was proved Bona et al. [1], for all $s \ge 1$.

With the global well-posedness result in hand, we can focus on the stabilization problem. For simplicity of notation we consider only the case

$$b_1 = b_2 = 1. \tag{1.3}$$

Then the conditions (1.2) take the simplified form

$$r = 0, \quad a_1^2 + a_2^2 = a_1 + a_2, \quad |a_3| < 1, \quad \text{and} \quad (a_1 - 1)a_3 = (a_2 - 1)a_3 = 0. \tag{1.4}$$

Hence either $a_3 = 0$ and $a_1^2 + a_2^2 = a_1 + a_2$, or $0 < |a_3| < 1$ and $a_1 = a_2 = 1$.

We prove the following theorem:

Theorem 1.2. *Assume* (1.3) *and* (1.4)*. If* $\phi, \psi \in H_p^s(0,1)$ *for some integer* $s \ge 3$*, then the solution of* (1.1) *satisfies the estimate*

$$\|u(t) - [u(t)]\|_{H_p^s(0,1)} + \|v(t) - [v(t)]\|_{H_p^s(0,1)} = o\left(e^{-k't}\right), \quad t \to \infty$$

for each $k' < k$.

An analogous theorem was proved in [10] for the usual KdV equation by using the infinite family of conservation laws for this equation. Such conservations lead to the construction of a suitable Lyapunov function that gives the exponential decay of the solutions. Here, we follow the same approach making use of the results established by Dávila and Chavez [7]. They proved that under the assumptions (1.2) system (1.1) also has an infinite family of conservation laws, and they conjectured the above theorem for this case. At this point we observe that some computations are simplified if we change u, v, ϕ and ψ to $u - [u]$, $v - [v]$, $\phi - [\phi]$ and $\psi - [\psi]$. Then, the new unknown functions u and v satisfy the same system (1.1) with ku and kv instead of $k(u-[u])$ and $k(v-[v])$. Hence we consider the solutions of the simplified system

$$\begin{cases} u' + uu_x + u_{xxx} + a_3 v_{xxx} + a_1 vv_x + a_2(uv)_x + ku = 0, \\ v' + vv_x + v_{xxx} + a_3 u_{xxx} + a_2 uu_x + a_1(uv)_x + kv = 0, \\ u(0,x) = \phi(x), \\ v(0,x) = \psi(x) \end{cases} \tag{1.5}$$

with periodic boundary conditions, corresponding to initial data ϕ, ψ with zero mean values.

In order to obtain the result, we prove a number of identities and estimates for the solutions of (1.1). In view of Theorem 1.1 it suffices to establish these estimates for *smooth solutions*, i.e., to solutions corresponding to C^∞ initial data ϕ, ψ with periodic boundary conditions. For such solutions all formal manipulations in the sequel will be justified.

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