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HYPERREFLEXIVITY CONSTANTS — A NUMERICAL EXPERIMENT

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ABSTRACT. We continue the study of hyperreflexivity constants of 1-dimensional subspaces of linear operators acting in $\ell^1_n(\mathbb{C})$ spaces [1, 3].

1. INTRODUCTION

Let m, n be positive integers and let $\mathbb{C}^{m\times n}$ be the space of all $m \times n$ complex matrices. Here we shall consider only the case $m = n = 2$. Matrices will be considered as linear operators acting in $\mathbb{C}^{2\times 1}$ equipped with ℓ^1 norm.

The *reflexive cover* of a linear subspace $\mathcal{S} \subseteq \mathbb{C}^{m\times n}$ is given by

$$\operatorname{Ref} \mathcal{S} = \{A \in \mathbb{C}^{m\times n};\ Ax \in Sx,\ \forall\ x \in \mathbb{C}^n\}\,.$$

Obviously $\mathcal{S} \subset \operatorname{Ref} \mathcal{S}$. If $S = \operatorname{Ref} S$, then S is said to be *reflexive*. It is well-known that every one-dimensional subspace is reflexive. Since we are working in finite dimensional spaces every reflexive space is also *hyperreflexive*, i.e. there exists the (minimal) constant $\kappa(\mathcal{S}) \geq 1$ such that $\alpha(A, \mathcal{S}) \leq \kappa(\mathcal{S})\ \operatorname{dist}(A, \mathcal{S})$ for every $A \in \mathbb{C}^{m\times n}$. Here

$$\operatorname{dist}(A, \mathcal{S}) = \min_{S\in\mathcal{S}} \|A - S\| = \min_{S\in\mathcal{S}} \max_{\|x\|=1} \|(A - S)x\|$$

denotes the usual and

$$\alpha(A, \mathcal{S}) = \max_{\|x\|=1} \min_{S\in\mathcal{S}} \|(A - S)x\|\,.$$

the Arveson distance of A to subspace $\mathcal{S}$.

In this paper we investigate the hyperreflexivity constant of the spaces $\mathcal{S}_k = \{\lambda J_k : \lambda \in \mathbb{C}\}$, where $k \geq 0$ and $J_k = \left(\begin{smallmatrix} 1 & 0 \\ 0 & k \end{smallmatrix}\right)$. It was shown in [1] that $\kappa(\mathcal{S}_0) = 1$ and $\kappa(\mathcal{S}_1) = \sqrt{2}$.

In Section 2 we present some results from [3] and a new lemma that allows to perform numerical experiments. In the last section we describe the results of a numerical experiment. The computations were made using MATLAB.

2. THEORETICAL RESULTS

The following lemma shows that it is enough to deal with spaces $\mathcal{S}_k$ for $k \geq 1$.

Lemma 1. *Let* $k \in (0, \infty)$, $J_k = \left(\begin{smallmatrix} 1 & 0 \\ 0 & k \end{smallmatrix}\right)$, *and* $\tilde{J}_k = \left(\begin{smallmatrix} k & 0 \\ 0 & 1 \end{smallmatrix}\right)$. *Put* $\mathcal{S}_k = \{\lambda J_k : \lambda \in \mathbb{C}\}$, $\tilde{\mathcal{S}}_k = \{\lambda \tilde{J}_k : \lambda \in \mathbb{C}\}$.

For every matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathbb{C}^{2\times 2}$ *and every vector* $u = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ *with* $\|u\|_1 = |x| + |y| = 1$ *it holds*

$$\operatorname{dist}(A, \lambda J_k) = \operatorname{dist}(\tilde{A}, \lambda \tilde{J}_k)\,, \tag{1}$$

$$\alpha(A, \lambda J_k) = \alpha(\tilde{A}, \lambda \tilde{J}_k)\,, \tag{2}$$

$$\kappa(\mathcal{S}_k) = \kappa(\mathcal{S}_{1/k}) \tag{3}$$

where $\tilde{A} = \left(\begin{smallmatrix} d & c \\ b & a \end{smallmatrix}\right)$.

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Proof. (1) is an obvious consequence of the equality $\|A - \lambda J_k\| = \|\tilde{A} - \lambda \tilde{J}_k\|$. Assertion (2) follows from the similar equality $\left\|(A - \lambda J_k)(x\ \ y)^\top\right\|_1 = \left\|(\tilde{A} - \lambda \tilde{J}_k)(y\ \ x)^\top\right\|_1$ and consequently $\kappa(\mathcal{S}_k) = \kappa(\tilde{\mathcal{S}}_k)$.

Now, putting $\mu = k\lambda$ we obtain for every $A \in \mathbb{C}^{2\times 2}$

$$A - \lambda J_k = A - \mu \tilde{J}_{1/k}$$

which shows that $\kappa(J_k) = \kappa(\tilde{J}_{1/k}) = \kappa(J_{1/k})$. □

The following result is a particular case of [3, Lemma 3.1] (see also [1] for the case $k = 1$).

Lemma 2. *Let* $k > 0$ *and* $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. *Then*

$$\operatorname{dist}(A, \mathcal{S}_k) = \max\left\{|b|, |c|, \tfrac{1}{1+k}\left(|ka - d| + |b| + |kc|\right)\right\}. \tag{4}$$

Now we use Lemma 2 to further reduction of computations. It is obvious that

$$\kappa(\mathcal{S}_k) = \sup\left\{\frac{\operatorname{dist}(A, \mathcal{S}_k)}{\alpha(A, \mathcal{S}_k)} \,:\, A \in \mathbb{C}^{2\times 2}\,,\ A \notin \mathcal{S}_k\right\}. \tag{5}$$

Since both functions $A \to \operatorname{dist}(A, \mathcal{S}_k)$ and $A \to \alpha(A, \mathcal{S}_k)$ are seminors we may and do assume that $\operatorname{dist}(A, \mathcal{S}_k) = 1$. Moreover, it is easy to see that we may also assume that $(2,2)$-entry of A is zero. More precisely [3, Lemma 3.2]

$$\kappa(\mathcal{S}_k) = \sup_{A\in\mathcal{A}} \frac{1}{\alpha(A, \mathcal{S}_k)}\,, \tag{6}$$

where

$$\mathcal{A} = \left\{\left(\begin{smallmatrix} a & be^{i\beta} \\ ce^{i\gamma} & 0 \end{smallmatrix}\right) : 1 > b, c \geq 0, b + c > 0, a = \frac{1}{k}(1 + k - b - kc),\ \beta, \gamma \in (-\pi, \pi]\right\}.$$

Finally, to compute $\max_{\|x\|=1} \min_{\lambda\in\mathbb{C}} \|(A - \lambda J_k)x\|$ for $A \in \mathcal{A}$ it is enough to take $x = (r, (1-r)e^{i\phi})^\top$, $0 < r < 1$, $\phi \in (-\pi, \pi]$. In this case, by [3, Theorem 3.3],

$$\min_{\lambda\in\mathbb{C}} \|(A - \lambda J_k)x\| = \min\{r, k(1-r)\}\left|a + b\frac{(1-r)}{r}e^{i\phi} - \frac{cr}{k(1-r)}e^{-i\phi}\right|. \tag{7}$$

3. A numerical experiment

Relations (6) and (7) can be used to write a MATLAB program which computes $\kappa(\mathcal{S}_k)$ (See Appendix). The numerical results are summarized in Tab. 1. Based on those computations we formulate

Conjecture 3. $\kappa(\mathcal{S}_k) \leq \sqrt{2}$ for all $k > 0$ and $\lim_{k\to\infty} \kappa(\mathcal{S}_k) = \lim_{k\to 0} \kappa(\mathcal{S}_k) = 1$.

Remark 4. It is known that in Hilbert space 1-dimensional subspaces have hyperreflexivity constant equal to 1 [2]. Using inequalities

$$\|x\|_2 = \sqrt{x_1^2 + x_2^2} \leq \|x\|_1 = |x_1| + |x_2|\,, \quad \|x\|_1 \leq \sqrt{2}\|x\|_2$$

it is easy to prove that $1 \leq \kappa(\mathcal{S}_k) \leq 2$ for all $k > 0$. 1-dimensional subspaces having $\kappa(\mathcal{S}) = 1$ are known [1, Theorem 1]. So it is natural to ask whether there exist a 1-dimensional subspace $\mathcal{S}$ of 2×2 matrices having $\kappa(\mathcal{S}) = 2$.

Table 1. Results of computing $\kappa(\mathcal{S}_k)$

k	1	1.2	1.4	1.6	1.8	2	2.2	2.4	2.6	2.8	3
$\kappa(\mathcal{S}_k)$	1.41421	1.41274	1.41342	1.41354	1.41406	1.41421	1.40985	1.40465	1.39984	1.39247	1.38442
k	3.5	4	4.5	5	6	7	8	9	10	11	12
$\kappa(\mathcal{S}_k)$	1.36379	1.34164	1.31864	1.29532	1.25000	1.24998	1.21622	1.19048	1.17021	1.15385	1.14035
k	13	14	15	20	25	30	35	40	50	100	200
$\kappa(\mathcal{S}_k)$	1.12903	1.11940	1.11111	1.093750	1.07438	1.06164	1.05263	1.04592	1.03659	1.01815	1.00903

References

[1] J. Bračič, V. Rozborová, M. Zajac, *Hyperreflexivity constants of some spaces of matrices*, Linear Algebra Appl. **439** (2013), 1340–1349

[2] B. Magajna, *On the distance to finite-dimensional subspaces in operator algebra*, J. London Math. Soc. **47** (1993), 516–532.

[3] V. Rozborová, *Hyperreflexivity constants of spaces of matrices*, Proceedings of the 15th Conference ELITECH'13, Faculty of Electrical Engineering and Information Technology STU, Bratislava, June 2013.

Appendix — MATLAB program.

```
clc; clear all
for k=1:0.2:2; %computes κ(S_k) for k=1, 1.2, ..., 1.9, 2
    r1=(k/(k+1)); h1=(r1-0.01)/10; h2=(1-r1)/10; %steps for x_1
    h=0.1; %step for entries of matrix A
    m2=1; %initializing of minimal α(A,S_k)
    for b1=0.1:h:1 %abs hodnota A[12]
        for c1=0:h:0.9; %abs. value of A[21]
        m1=max([c1 b1]); %first estimate of α(A,S_k) (for r=1,r=0)
        a=((1-b1)/k)+(1-c1); %A[11]
            for t1=-1:h:1-h; b=b1*exp(i*t1*pi); %A[12]/π
                for t2=-1:h:1-h; c=c1*exp(i*t2*pi); %A[21]
                    for r=0.01:h1:r1-h1; %x1
                        for t3=-1:0.05:0.95; %(arg x2)/π
                        x1=r; x2=(1-r)*exp(i*t3*pi); %unit vector x=(x1,x2)
                        ALPHAx=(abs(a*x1+b*x2-(c*r*r)/(k*x2)));
                        m1=max([m1 ALPHAx]);
                        end
                    end
                    for r=r1:h2:1-h2;
                        for t3=-1:0.05:0.95; %arg x2/pi
                        x1=r; x2=(1-r)*exp(i*t3*pi); %jednotkovy vektor x
                        ALPHAx=abs((k*x2*a)+((b*k*x2*x2)/x1)-(c*x1));
                        end
                    end
                    m2=min([m2 m1]);
                end
            end
        end
    end
k kappa=1/m2 %output: k, κ(S_k), i.e., actual k and kappa(S_k) are displayed
end
```

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