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SUBSETS OF BOUNDED POSITIVE OPERATORS MAY FORM GENERALIZED EFFECT ALGEBRAS

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1. Introduction and preliminaries

This contribution is based on the work [3].

Effect algebra is a structure introduced by Foulis and Bennett in [2]. The prototype of this structure is the set of Hilbert space effects $\mathcal{E}(\mathcal{H}) = \{A \in \mathcal{B}(\mathcal{H}) \mid 0 \leq A \leq I\}$. The corresponding effect algebraic operation $\oplus$ is (a restriction of) the usual addition of operators.

Let $\mathcal{H}$ be an infinite-dimensional separable complex Hilbert space. Let $\mathcal{B}(\mathcal{H})$ denote the set of all bounded linear operators on $\mathcal{H}$, $\mathcal{B}_p(\mathcal{H})$ is the set of all bounded positive linear operators on $\mathcal{H}$. The set of all compact operators on $\mathcal{H}$ is denoted by $\mathrm{Com}(\mathcal{H})$, the set of all positive compact operators on $\mathcal{H}$ is denoted by $\mathrm{Com}_p(\mathcal{H})$. The set of all trace-class operators on $\mathcal{H}$ is denoted by $\mathcal{J}_1(\mathcal{H})$, the set of all positive trace-class operators on $\mathcal{H}$ is denoted by $\mathcal{J}_{1p}(\mathcal{H})$. The set of all Hilbert-Schmidt operators on $\mathcal{H}$ is denoted by $\mathcal{J}_2(\mathcal{H})$, the set of all positive Hilbert-Schmidt operators on $\mathcal{H}$ is denoted by $\mathcal{J}_{2p}(\mathcal{H})$.

Now we give some definitions and basic facts about quantum structures.

Definition 1 (Foulis and Bennett, [2])**.** A partial algebra $(E; \oplus, 0, 1)$ is called an *effect algebra* if 0,1 are two distinct elements and $\oplus$ is a partially defined binary operation on E which satisfy the following conditions for any $x, y, z \in E$:

(E1) $x \oplus y = y \oplus x$ if $x \oplus y$ is defined,
(E2) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ if one side is defined,
(E3) for every $x \in E$ there exists a unique $y \in E$ such that $x \oplus y = 1$ (we put $x' = y$),
(E4) If $1 \oplus x$ is defined then $x = 0$.

Definition 2. Let E be an effect algebra. Then $Q \subseteq E$ is called a *sub-effect algebra* of E if

(i) $1 \in Q$,
(ii) if out of elements $x, y, z \in E$ with $x \oplus y = z$ two are in Q, then $x, y, z \in Q$.

Definition 3. (1) A *generalized effect algebra* $(E; \oplus, 0)$ is a set E with an element $0 \in E$ and partial binary operation $\oplus$ satisfying for any $x, y, z \in E$ conditions

(GE1) $x \oplus y = y \oplus x$ if one side is defined,
(GE2) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ if one side is defined,
(GE3) If $x \oplus y = x \oplus z$ then $y = z$,
(GE4) If $x \oplus y = 0$ then $x = y = 0$,
(GE5) $x \oplus 0 = x$ for all $x \in E$.

(2) Define a binary relation $\leq$ on E by

$x \leq y$ iff for some $z \in E$, $x \oplus z = y$.

(3) $Q \subseteq E$ is called a *sub-generalized effect algebra* of E if and only if out of elements $x, y, z \in E$ with $x \oplus y = z$ at least two are in Q then $x, y, z \in Q$.

Note that every sub-generalized effect algebra of E is a generalized effect algebra in its own right.

The well known facts are that the relation $\leq$ in Definition 3 is a partial order on E for which 0 is the least element of E. Moreover, E is an effect algebra iff E has a greatest element 1 .

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Further, if $(E;\oplus,0,1)$ is an effect algebra, then $(E;\oplus,0)$ is a generalized effect algebra (with the same operation $\oplus$) (see [1]).

Assume that $(E;\oplus,0)$ is a generalized effect algebra. Then (see, e.g., [5]) for any fixed $q\in E$, $q\neq 0$ the interval

$$[0,q]_E=\{x\in E \mid \text{there exists } y\in E \text{ with } x\oplus y=q\}$$

is an effect algebra $([0,q]_E;\oplus_q,0,q)$ with unit q and with the partial operation $\oplus_q$ defined for $x,y\in[0,q]_E$ by

$$x\oplus_q y \text{ exists and } x\oplus_q y=x\oplus y \text{ iff } x\oplus y\in[0,q]_E \text{ exists in } E.$$

Then, as it follows from the previous, $([0,q]_E;\oplus_q,0)$ is a generalized effect algebra in the natural sense.

Let us define the operation $\oplus$ on $\mathcal{B}_p(\mathcal{H})$ by $A\oplus B=A+B$ (the usual sum of operators) and let 0 denote the null operator. The next Proposition is a straightforward consequence of Theorem 3.5 in [4].

Proposition 4. *$(\mathcal{B}_p(\mathcal{H});\oplus,0)$ is a generalized effect algebra and $\oplus$ is a total operation.*

A standard partial ordering $\leq$ on $\mathcal{B}_p(\mathcal{H})$ is defined in following way:

$$A\leq B \text{ iff } B-A \text{ is a positive operator.}$$

Let us consider the set of Hilbert space effects

$$\mathcal{E}(\mathcal{H})=\{A\in\mathcal{B}_p(\mathcal{H}) \mid 0\leq A\leq I\}$$

where I is the identity operator. It is an effect algebra [2] $(\mathcal{E}(\mathcal{H});\oplus_I,0,I)$ with the partial effect-algebraic operation $\oplus_I$ being the restriction of the usual sum of operators to the set $\mathcal{E}(\mathcal{H})$ (for $A,B\in\mathcal{E}(\mathcal{H}), A\oplus_I B=A+B$ is defined iff $A+B\in\mathcal{E}(\mathcal{H})$). Consequently, it is a generalized effect algebra $([0,I]_{\mathcal{B}_p(\mathcal{H})};\oplus_I,0)$. (Of course, $\mathcal{E}(\mathcal{H})=[0,I]_{\mathcal{B}_p(\mathcal{H})}$.)

In the present paper, we show more examples of generalized effect algebras of bounded operators on Hilbert space. They are positive compact operators, positive Hilbert-Schmidt operators and positive trace-class operators. All these are sub-generalized effect algebras of the generalized effect algebra of bounded positive operators (with the $\oplus$ operation being the usual addition of operators). Moreover, three nontrivial sub-generalized effect algebras of $\mathcal{E}(\mathcal{H})$ (regarded as a generalized effect algebra) are shown. They are the sets of compact, Hilbert-Schmidt and trace-class operators intersected with the set $\mathcal{E}(\mathcal{H})$. These results are stated also in a more general form.

2. Results

Theorem 5. *Let $\mathcal{L}$ be a linear subspace of $\mathcal{B}(\mathcal{H})$. Then the set $\mathcal{L}_p=\mathcal{L}\cap\mathcal{B}_p(\mathcal{H})$ of positive operators in $\mathcal{L}$ is a sub-generalized effect algebra of $(\mathcal{B}_p(\mathcal{H});\oplus,0)$ with respect to inherited $\oplus$-operation. Hence it is a generalized effect algebra in its own right.*

Corollary 6. *The sets $\mathrm{Com}_p(\mathcal{H}),\mathcal{J}_{2p}(\mathcal{H})$ and $\mathcal{J}_{1p}(\mathcal{H})$ are sub-generalized effect algebras of $(\mathcal{B}_p(\mathcal{H});\oplus,0)$ with respect to inherited $\oplus$-operations. Hence they are generalized effect algebras in their own right.*

Let

$$\mathcal{E}_{\mathrm{Com}}(\mathcal{H})=[0,I]_{\mathcal{B}_p(\mathcal{H})}\cap\mathrm{Com}(\mathcal{H}),$$
$$\mathcal{E}_{\mathcal{J}_2}(\mathcal{H})=[0,I]_{\mathcal{B}_p(\mathcal{H})}\cap\mathcal{J}_2(\mathcal{H}),$$
$$\mathcal{E}_{\mathcal{J}_1}(\mathcal{H})=[0,I]_{\mathcal{B}_p(\mathcal{H})}\cap\mathcal{J}_1(\mathcal{H}).$$

All of them are special cases of

$$\mathcal{E}_{\mathcal{L}}(\mathcal{H})=[0,I]_{\mathcal{B}_p(\mathcal{H})}\cap\mathcal{L}$$

where $\mathcal{L}$ is an arbitrary linear subspace of $\mathcal{B}(\mathcal{H})$.

Theorem 7. *Let $\mathcal{L}$ be a linear subspace of $\mathcal{B}(\mathcal{H})$. Then the set $\mathcal{E}_{\mathcal{L}}(\mathcal{H})$ is a sub-generalized effect algebra of the generalized effect algebra $\mathcal{E}(\mathcal{H})$. In particular, each of the sets $\mathcal{E}_{\mathcal{J}_1}(\mathcal{H})$, $\mathcal{E}_{\mathcal{J}_2}(\mathcal{H})$, $\mathcal{E}_{\mathrm{Com}}(\mathcal{H})$ form a (nontrivial) sub-generalized effect algebra of $\mathcal{E}(\mathcal{H})$. Moreover,*

$$\emptyset \neq \mathcal{E}_{\mathcal{J}_1}(\mathcal{H}) \subsetneqq \mathcal{E}_{\mathcal{J}_2}(\mathcal{H}) \subsetneqq \mathcal{E}_{\mathrm{Com}}(\mathcal{H}) \subsetneqq \mathcal{E}(\mathcal{H}).$$

Remark. As $\mathcal{J}_1(\mathcal{H}) \subseteq \mathcal{J}_2(\mathcal{H}) \subseteq \mathrm{Com}(\mathcal{H})$ and it is well-known that $I \notin \mathrm{Com}(\mathcal{H})$ for infinite-dimensional $\mathcal{H}$, we have $I \notin \mathcal{E}_{\mathrm{Com}}(\mathcal{H}), I \notin \mathcal{E}_{\mathcal{J}_2}(\mathcal{H}), I \notin \mathcal{E}_{\mathcal{J}_1}(\mathcal{H})$, so the sets $\mathcal{E}_{\mathrm{Com}}(\mathcal{H})$, $\mathcal{E}_{\mathcal{J}_2}(\mathcal{H})$, $\mathcal{E}_{\mathcal{J}_1}(\mathcal{H})$ are not sub-effect algebras of $\mathcal{E}(\mathcal{H})$ as I is the top element of the effect algebra $\mathcal{E}(\mathcal{H})$ (see Definition 2 (i)).

As $(\mathcal{B}_p(\mathcal{H}); \oplus, 0)$ is a generalized effect algebra, for arbitrary $A \in \mathcal{B}_p(\mathcal{H})$, $A \neq 0$ the interval $[0, A]_{\mathcal{B}_p(\mathcal{H})}$ is an effect algebra which is a generalized effect algebra in the natural sense. A special example is the choice $A = I$ when we get the interval $\mathcal{E}(\mathcal{H}) = [0, I]_{\mathcal{B}_p(\mathcal{H})}$. For $\mathcal{L}$ being an arbitrary linear subspace of $\mathcal{B}(\mathcal{H})$ we have that $\mathcal{E}_{\mathcal{L}}(\mathcal{H})$ is a sub-generalized effect algebra of $\mathcal{E}(\mathcal{H})$. The same argument shows that for $A \in \mathcal{B}_p(\mathcal{H}), A \neq 0$ being arbitrary, the set

$$\mathcal{E}_{A\mathcal{L}}(\mathcal{H}) = [0, A]_{\mathcal{B}_p(\mathcal{H})} \cap \mathcal{L}$$

is a sub-generalized effect algebra of the generalized effect algebra $\mathcal{E}_A(\mathcal{H}) = [0, A]_{\mathcal{B}_p(\mathcal{H})}$. If we specify $\mathcal{L}$ to be one of the sets $\mathcal{J}_1(\mathcal{H})$, $\mathcal{J}_2(\mathcal{H})$, $\mathrm{Com}(\mathcal{H})$, we conclude that the sets

$$\mathcal{E}_{A\mathcal{J}_1}(\mathcal{H}) = [0, A]_{\mathcal{B}_p(\mathcal{H})} \cap \mathcal{J}_1(\mathcal{H}), \tag{1}$$

$$\mathcal{E}_{A\mathcal{J}_2}(\mathcal{H}) = [0, A]_{\mathcal{B}_p(\mathcal{H})} \cap \mathcal{J}_2(\mathcal{H}) \tag{2}$$

and

$$\mathcal{E}_{A\mathrm{Com}}(\mathcal{H}) = [0, A]_{\mathcal{B}_p(\mathcal{H})} \cap \mathrm{Com}(\mathcal{H})$$

are sub-generalized effect algebras of $\mathcal{E}_A(\mathcal{H})$. In the special case $A = I$, we have that these sets $(\mathcal{E}_{A\mathcal{J}_1}(\mathcal{H}) = \mathcal{E}_{\mathcal{J}_1}(\mathcal{H}), \mathcal{E}_{A\mathcal{J}_2}(\mathcal{H}) = \mathcal{E}_{\mathcal{J}_2}(\mathcal{H}), \mathcal{E}_{A\mathrm{Com}}(\mathcal{H}) = \mathcal{E}_{\mathrm{Com}}(\mathcal{H}))$ are mutually different and different from $\mathcal{E}(\mathcal{H}) = \mathcal{E}_I(\mathcal{H})$. There arises a natural question if this is satisfied also for general $A \neq I$. The following Theorem shows that for arbitrary positive $A \in \mathcal{J}_1(\mathcal{H})$ the answer is negative.

Theorem 8. *Let $A \in \mathcal{B}_p(\mathcal{H}) \cap \mathcal{J}_1(\mathcal{H})$. Then*

$$\mathcal{E}_{A\mathcal{J}_1}(\mathcal{H}) = \mathcal{E}_{A\mathcal{J}_2}(\mathcal{H}) = \mathcal{E}_{A\mathrm{Com}}(\mathcal{H}) = \mathcal{E}_A(\mathcal{H}).$$

References

[1] A. Dvurečenskij, S. Pulmannová, *New Trends in Quantum Structures*, Kluwer, Dordrecht, 2000.
[2] D.J. Foulis, M.K. Bennett, Effect algebras and unsharp quantum logics, Found. Phys. **24** (1994), 1331–1352.
[3] M. Polakovič, Generalized effect algebras of bounded positive operators defined on Hilbert spaces, Rep. Math. Phys. **68** (2011), 241–250.
[4] M. Polakovič, Z. Riečanová, Generalized effect algebras of positive operators densely defined on Hilbert spaces, Int. J. Theor. Phys **50** (2011), 1167–1174.
[5] Z. Riečanová, Subalgebras, intervals and central elements of generalized effect algebras, Int. J. Theor. Phys. **39** (1999), 3209–3220.

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